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Twistor Theory for Riemannian Symmetric Spaces

With Applications to Harmonic Maps of Riemann Surfaces

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Introduction

Background

The subject of this monograph is the interaction between real and complex homogeneous geometry and its application to the study of minimal surfaces (or harmonic maps). That minimal surfaces may be studied by complex variable methods is by no means a new idea since it informs the work of Weierstrass on minimal surfaces in $\mathbf{R}^3$. However, we shall take as our starting point the seminal papers of Calabi [23, 24] in the late sixties. In those papers, Calabi investigated minimal immersions of a 2-sphere in S^{2n} by associating to each such immersion a holomorphic curve in the homogeneous Kähler manifold $\mathbf{SO}(2n+1)/\mathbf{U}(n)$. The methods of Complex Analysis were then brought to bear on these auxiliary holomorphic curves. The fruits of this analysis include a complete classification of all minimal 2-spheres in an n-sphere in terms of certain holomorphic 2-spheres in a complex projective space and a quantisation of the area of such minimal surfaces.

These ideas were taken up in the eighties by a number of physicists and mathematicians and a similar analysis of harmonic maps (i.e. branched minimal immersions) of a 2-sphere in a complex projective space was soon provided [12, 29, 34, 36]. Again, a key step is to associate to each such harmonic map a holomorphic map of S^2 into an auxiliary complex manifold, in this case a flag manifold of the form $\mathbf{U}(n+1)/\mathbf{U}(r)\times\mathbf{U}(1)\times\mathbf{U}(n-r)$. Since then there has been a great deal of activity in extending these results to other co-domains such as Grassmannians [1, 21, 22, 28, 59, 83], Lie groups [72, 73, 85] and other classical symmetric spaces [82, 3].

Meanwhile, in 1984, Eells-Salamon [33] observed that more flexibility could be obtained by considering pseudo-holomorphic curves in certain non-integrable almost complex manifolds. Indeed, they associated such a curve to any conformal harmonic map of a Riemann surface into an oriented Riemannian 4-manifold. Now pseudo-holomorphic curves are much less easy to handle (see, however, [39]) but, nonetheless, these ideas provided a useful framework in which many of the previous results could be understood.

Finally, in 1985, Uhlenbeck's analysis of harmonic 2-spheres in the unitary group $\mathbf{U}(n)$ appeared [72]. Here again, a decisive role is played by holomorphic curves in an auxiliary complex space but this time the space in question is the infinite-dimensional Kähler manifold of based loops in $\mathbf{U}(n)$: the loop group $\Omega\mathbf{U}(n)$. One of Uhlenbeck's main results is the existence of a Bäcklund transform, repeated application of which produces all harmonic 2-spheres in $\mathbf{U}(n)$ from the constant maps. This provides a classification of all such 2-spheres which subsumes and extends most of the known results in this direction for harmonic 2-spheres in complex Grassmannians.

Throughout a large part of the above development, attention has focussed on harmonic maps into certain Riemannian symmetric spaces together with holomorphic curves in associated homogeneous Kähler manifolds. This monograph has its genesis in our attempt to understand the relationship provided by these ideas between the geometry of symmetric spaces and the geometry of complex homogeneous spaces.

Before turning to a discussion of this relationship, we refer the Reader who wishes to learn more of the matters so briefly touched upon above to the surveys [16, 17, 84].

Overview

There are three main topics that are treated in this work: homogeneous geometry, twistor theory and harmonic maps. Let us describe each of these in turn:

Homogeneous Geometry

We deal with two classes of reductive homogeneous spaces: the Riemannian symmetric spaces and the (generalised) flag manifolds. The first class have been extensively studied and need no introduction here. The flag manifolds are comparatively less well-known (see, however, [5, 80]) although they have a rich geometry and exhaust the compact Kähler homogeneous spaces with semi-simple isometry group [6]. One of our main aims is to demonstrate a close relationship that exists between flag manifolds and the Riemannian symmetric spaces with inner involution (we call these *inner* symmetric spaces). To be more precise, we show that each flag manifold G/H fibres homogeneously over an inner symmetric space G/K in an essentially unique way. Moreover, each inner symmetric space is the target of such a fibration for at least one flag manifold (and generally more than one). These fibrations, which we call *canonical* fibrations, are defined entirely by the algebra of the situation: specifically, by the data of a parabolic subgroup of a complex semi-simple Lie group and a compact real form of that Lie group.

We shall also identify an invariant holomorphic distribution (the *superhorizontal* distribution) which is transverse to the fibres of our canonical fibrations. This distribution enjoys the property that holomorphic curves tangent to it project onto minimal surfaces under the canonical fibrations. In favourable circumstances, all minimal 2-spheres in the symmetric space arise in this way.

A useful tool in this development is a particular realisation of the flag manifold as an adjoint orbit (the orbit of the *canonical element*) which, although completely natural, appears to be quite different from other such realisations discussed by Borel [65].

It is perhaps surprising that the non-compact version of these ideas is rather better known. In fact, there is substantial overlap between the non-compact analogue of our theory and the theory of period matrix domains [38]. In particular, in that setting, the superhorizontal distribution is just that which defines the infinitesimal period relation.

These fibrations also appear in the thesis of W. Schmid [64] in his study of irreducible representations of non-compact semisimple Lie groups obtained on cohomology groups of holomorphic line bundles over flag domains.

Twistor theory

A central role in the twistor theory of a Riemannian manifold N is played by the bundle $J(N) \to N$ of almost Hermitian structures on N. This bundle carries a natural almost complex structure (denoted here by J_1) which is, however, rarely integrable. In a search for complex manifolds associated to N, we consider the zero set Z of the Nijenhuis tensor of J_1, which is *a priori* a set with very little structure. However, when N is an inner symmetric space, we shall show that the isometry group of N acts transitively on each connected component of Z and that each such component is in fact a flag manifold holomorphically embedded in $J(N)$. Moreover, the trace of the bundle projection on each component is a canonical fibration and all canonical fibrations are realised in this way exactly once. This gives a geometrical interpretation of the algebraic constructions discussed above and, at the same time, completely elucidates the structure of Z. It seems clear that this theory will have applications to the geometry of inner symmetric spaces. We shall present some preliminary results in this direction and refer the Reader also to

[19].

The structure theorem for Z is closely related to some results of Bryant [11] who considered the intersection of Z and the zero set of the obstruction to holomorphicity of the horizontal distribution of $J(N)$. This smaller zero set again has flag manifolds for components although of a rather restricted type (in the terminology of chapter 4, they have height not exceeding two). Thus our results may be viewed as an extension of those of Bryant. Our methods, however, are quite different and provide a new proof of Bryant's theorem.

Harmonic maps

As applications of our theory, we study harmonic maps of S^2 into a symmetric space N. Our results may be split into three categories.

Firstly, we show that if N is inner then any such harmonic map is the image under a canonical fibration of a pseudo-holomorphic curve in a flag manifold. Here the flag manifold is equipped with a non-integrable almost complex structure *à la* Eells-Salamon but we show that, under certain conditions, the vanishing of a holomorphic differential guarantees that the pseudo-holomorphic curves are in fact holomorphic for the standard complex structure on the flag manifold. This provides a uniform proof of results of Calabi [24], Eells-Wood [34] and Bryant [10] concerning harmonic 2-spheres in S^{2n}, $\mathbf{C}P^n$ and $\mathbf{H}P^1$ respectively. Much of the previous theory of minimal 2-spheres has depended on the vanishing of a series of holomorphic differentials. In our approach, we identify a universal holomorphic differential which is also of use in our study of stable harmonic 2-spheres, to which we now turn.

We completely characterise the stable harmonic 2-spheres in a simply-connected irreducible symmetric space of compact type N. It turns out that the results depend solely on $\pi_2(N)$ which is either zero, $\mathbf{Z}_2$ or $\mathbf{Z}$ (in case that N is Hermitian symmetric). If N is Hermitian symmetric, we obtain an *a priori* proof of the result that stable harmonic 2-spheres are $\pm$-holomorphic; a result originally proved by Siu-Zhong [66, 87] by checking cases. If $\pi_2(N)$ vanishes, we show that all stable harmonic 2-spheres are constant. The remaining case, when $\pi_2(N)=\mathbf{Z}_2$, is rather more interesting. here N contains a family of totally geodesically immersed Hermitian symmetric subspaces (which turn out to be projective spaces) with the property that maps factoring holomorphically through one are stable and harmonic as maps into N. Moreover, we show that any stable harmonic 2-sphere in N must so factor. It is an interesting corollary of this development that such a symmetric space must admit a null-homotopic stable harmonic 2-sphere which is non-constant. These results form joint work with Simon Salamon and we thank him for his permission to let us present them here.

Finally, we extend the results of Uhlenbeck [72] mentioned above to a large class of simple Lie groups G (those with Hermitian symmetric quotient). In so doing, we follow Valli's insightful approach [73] to Uhlenbeck's work. We find a Bäcklund transform for harmonic maps $S^2 \to G$ with which all such maps may be constructed from the constant maps. This answers to a large extent question 2 posed by Uhlenbeck in [72]. As an application of these ideas, we obtain a number of results concerning gap phenomena for harmonic 2-spheres in symmetric spaces some of which are new.

Remark on methods

The Reader may be dismayed but not surprised to learn that we have recourse to a large amount of Lie theory during this work. Perhaps more novel and interesting is our repeated use of the theorem of Birkhoff-Grothendieck on the decomposition of holomorphic vector bundles and the reduction of holomorphic principal bundles on S^2. Indeed, chapters 2 and 6–8 may be read as an essay on the applications of this theorem to the study of harmonic 2-spheres. This idea is not new: in the investigation of stable maps, its use may be traced back to Siu-Yau [67], while, in studies of the construction of harmonic maps, it makes its first fleeting appearance in the work of Erdem-Wood [35]. However, we hope to demonstrate in this work that its use as a powerful and unifying tool in this area has been underestimated. In particular, this appears to be the first time that the full force of the Grothendieck version of the theorem [40] has been applied to harmonic maps.

Table of contents

As a guide to the Reader, we present a brief description of the contents of each chapter.

Chapter 1 contains generalities about homogeneous geometry and sets up our approach to that subject. The main point of interest is that we define a Lie algebra valued 1-form on a reductive homogeneous space which satisfies an analogue of the Maurer-Cartan equations. Much of the geometry of such a space can be described using this 1-form and in this way we provide a framework for calculating on homogeneous spaces which is of great use in the sequel.

Chapter 2 introduces harmonic maps and we apply the Birkhoff-Grothendieck theorem for the first time to demonstrate the existence of pseudo-holomorphic curves in $J(N)$ covering a harmonic map $S^2 \to N$ where N is an even-dimensional manifold. Extensions of this result to suitably ramified minimal surfaces of higher genus in Kähler manifolds via the Harder-Narasimhan filtration are also presented.

Chapter 3 introduces symmetric spaces. Here also are collected various items of structure theory which we need in the sequel. The main result of the chapter is the determination of the second homotopy group of the irreducible symmetric spaces of compact type. We provide a simple root-theoretic criterion for determining this group together with an explicit set of generators. Our approach is based on Murakami's version [53] of the classification of involutions of a compact simple Lie algebra. The Reader with no taste for structure theory is invited to skip most of this.

Chapter 4 is one of the most central in the monograph. After some algebraic preliminaries, we define flag manifolds and their non-compact analogues, the flag domains. We then construct the canonical fibrations and superhorizontal distributions and give some examples. Our construction throws up a particular choice of Kähler metric on a flag manifold and we discuss the cohomology class that the Kähler form represents. This last will be needed in chapter 8. Finally, an appendix describes the adjustments to be made for non-inner symmetric spaces.

Chapter 5 is consecrated to the zero set of the Nijenhuis tensor on $J(N)$ and the determination of its structure.

Chapter 6 describes the covering of harmonic 2-spheres in symmetric spaces by pseudo-holomorphic curves in flag manifolds. Just as in chapter 2 we use the Birkhoff-Grothendieck theorem for this. We also discuss holomorphic differentials.

Chapter 7 contains our classification of stable harmonic 2-spheres in compact symmetric spaces.

Chapter 8 is concerned with our Bäcklund transform procedure for harmonic 2-spheres in a simple Lie group and its applications.

Final remarks and acknowledgments

Some of the work described in chapters 4, 6 and 8 was announced in [18] while the results of chapter 7 were announced in [20]. Readers with long memories should also note that a preliminary version of this manuscript was in circulation under the title *Twistors, Homogeneous*

Geometry and Harmonic Maps.

During the lengthy course of the preparation of this monograph, we have benefited from innumerable conversations with too many colleagues to mention. Special thanks are due to Simon Salamon, especially for his collaboration in the results of chapter 7; Georgio Valli, for informing us of his work and John C. Wood and Jim Eells for their advice and encouragement.

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Chapter 1. Homogeneous Geometry

Let M be a manifold on which is given a smooth transitive action of a Lie group G. Choosing a base-point $x_0 \in M$ we let H be the stability subgroup of x_0 and then we have a principal H-bundle $\pi : G \to M$, where $\pi(g) = g.x_0$, on which G acts by left translations as bundle automorphisms. Indeed M is diffeomorphic to G/H and then π is just the coset fibration. The surjective map of the Lie algebra $\mathbf{g}$ of G, $\mathbf{g} \to T_{x_0}M$ given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp t\xi.x_0$$

has the Lie algebra $\mathbf{h}$ of H as kernel and so induces an isomorphism of $\mathbf{g}/\mathbf{h}$ with $T_{x_0}M$. We extend this by equivariance to get an isomorphism of the associated bundle $G\times_H \mathbf{g}/\mathbf{h}$ with TM which is given explicitly by

$$[g,\xi+\mathbf{h}] \mapsto g_* \left.\frac{d}{dt}\right|_{t=0} \exp t\xi.x_0 = \left.\frac{d}{dt}\right|_{t=0} \exp t\mathrm{Ad}g\xi.x$$

where $x = \pi(g)$.

Notation. If W is a representation of H we shall henceforth denote the associated bundle $G\times_H W$ by $[W]$.

The homogeneous space M is said to be *reductive* if the Lie algebra $\mathbf{g}$ has a decomposition $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ for some $\mathrm{Ad}_G H$-invariant summand $\mathbf{m}$. Then $\mathbf{g}/\mathbf{h} \cong \mathbf{m}$ as H-spaces so $[\mathbf{g}/\mathbf{h}] \cong [\mathbf{m}]$ and hence we have an isomorphism $[\mathbf{m}] \cong TM$. Since $\mathbf{m}$ is an invariant subspace of $\mathbf{g}$ we have an inclusion $[\mathbf{m}] \subset [\mathbf{g}]$. The latter may be canonically identified with the trivial bundle $\underline{\mathbf{g}} = M \times \mathbf{g}$ via

$$[g,\xi] \mapsto (\pi(g), \mathrm{Ad}g\xi).$$

Thus we have an identification of TM with a subbundle of the trivial bundle which we may view as a $\mathbf{g}$-valued 1-form β on M.

If $P_{\mathbf{m}}$ (resp. $P_{\mathbf{h}}$) denotes the projection onto $\mathbf{m}$ (resp. $\mathbf{h}$) then we have

$$\beta_x(X) = \mathrm{Ad}g P_{\mathbf{m}}(\mathrm{Ad}g^{-1}\xi) \quad \text{if} \quad X = \left.\frac{d}{dt}\right|_{t=0} \exp t\xi.x \tag{1}$$

and $x = \pi(g)$. From this it is easy to see that β is equivariant in the sense that

$$g^*\beta = \mathrm{Ad}g\beta.$$

Note the useful identity

$$X = \left.\frac{d}{dt}\right|_{t=0} \exp t\beta_x(X).x, \quad X \in T_xM.$$

Example. If M is actually the group manifold G, acting on itself by left translations, then we see from the above identity that β is just the (right) Maurer-Cartan form of G.

By analogy with the above example, we shall call β the *Maurer-Cartan form* of the reductive homogeneous space M.

A homogeneous space M of G is called a *symmetric* space if there is an involution τ_0 of G with $(G^{\tau_0})_0 \subset H \subset G^{\tau_0}$. Then $\mathbf{h}=\mathbf{g}^{\tau_0}$ and $\mathbf{m}=\mathbf{g}^{-\tau_0}$ is a reductive summand which satisfies the additional condition $[\mathbf{m},\mathbf{m}] \subset \mathbf{h}$. Symmetric spaces are the most widely studied class of reductive homogeneous spaces. We shall see the geometrical significance of the extra condition on the reductive summand in this special case shortly.

Returning to the general situation, the left translation of a reductive summand $\mathbf{m}$ around G provides a G-invariant distribution which is horizontal for π and right H-invariant. This defines a G-invariant connection in the principal bundle $\pi : G \to M$. This procedure produces a bijective correspondence between reductive summands $\mathbf{m}$ and G-invariant connections in $\pi : G \to M$. We shall view the reductive homogeneous space M as coming equipped with a fixed summand $\mathbf{m}$ and refer to the corresponding connection as the *canonical connection*. Its connection form α (as an $\mathbf{h}$-valued 1-form on G) is $P_{\mathbf{h}}\theta$ where θ is the left-invariant Maurer-Cartan form of G. G-invariant tensors on M (or, more generally, G-invariant sections of associated bundles) are parallel with respect to the canonical connection and in particular the canonical connection is a metric connection for any invariant metric on M. Given such a metric, TM becomes isomorphic with the cotangent bundle T^*M and so is a symplectic G-space. β is essentially the momentum map for this symplectic action of G.

The canonical connection induces a covariant differentiation D in any associated bundle $[V]$ for any representation V of H. If V happens to be the restriction of a representation of G then $[V]$ can be identified with the trivial bundle $\underline{V}$ via the map

$$[g,v] \mapsto (\pi(g), g.v).$$

In this case there is a simple relationship between flat differentiation and the covariant differentiation induced on $\underline{V}$ by the canonical connection.

Proposition 1.1. *Let $f : M \to \underline{V}$ be a smooth section of $\underline{V}$ then*

$$df = Df + \beta.f \quad .$$

Proof. Under the identification of $\underline{V}$ with $[V]$ a section f of $\underline{V}$ corresponds with an H-equivariant map $\hat{f} : G \to V$ as follows

$$\hat{f}(g) = g^{-1}.f(\pi(g)) \quad . \tag{2}$$

Further, the covariant differential Df lifts as the V-valued 1-form $d\hat{f}+\alpha.\hat{f}$ on G. Differentiating (2) and using the Leibnitz rule gives

$$d\hat{f}_g = -\theta.g^{-1}.f \circ \pi + g^{-1}.\pi^* df \quad .$$

Thus

$$\begin{aligned}(d\hat{f}+\alpha.\hat{f})_g &= g^{-1}.\pi^* df + (\alpha-\theta)_g.\hat{f} \\ &= g^{-1}.\pi^* df - P_{\mathbf{m}}\theta.\hat{f} \\ &= g^{-1}(\pi^* df - \mathrm{Ad}g(P_{\mathbf{m}}\theta).f \circ \pi).\end{aligned}$$

Now, pulling equation (1) back to G we have

$$(\pi^*\beta)_g = \mathrm{Ad}g(P_{\mathbf{m}}\theta)$$

whence

$$d\hat{f} + \alpha.\hat{f} = g^{-1}.\pi^*(df - \beta.f)$$

and so

$$Df = df - \beta.f$$

concluding the proof. □

Suppose W is an H-invariant subspace of the G-representation V. Then $[W]$ is a G-invariant subbundle of the trivial bundle $\underline{V}$ and hence the inclusion $[W] \subset \underline{V}$ is parallel for the canonical connection. Thus one way to view proposition (1.1) is as a formula for the canonical connection in $[W]$. If $V = W_1 + W_2$ is an H-invariant splitting of a representation of G then

$$\underline{V} = [W_1] + [W_2]$$

is a G-invariant splitting of $\underline{V}$ into D-parallel subbundles. As a simple consequence of proposition (1.1) we have the following:

Lemma 1.2 *Denote by $\sigma : G \to End(V)$ the representation of G on V and let $\underline{V} = [W_1] + [W_2]$. Let $P_i : V \to [W_i]$ be the projection onto $[W_i]$ viewed as a function $P_i : M \to End(V)$. Then*

$$dP_i = [\sigma(\beta), P_i].$$

Proof. The projections are G-invariant and hence D-parallel. But on $\underline{\mathrm{End}(V)}$

$$D = d - [\sigma(\beta), .],$$

whence the result. □

Example. Letting σ be the adjoint representation of G we see that

$$\underline{\mathbf{g}} = [\mathbf{h}] + [\mathbf{m}] \quad .$$

As we remarked above $[\mathbf{m}] = TM$ while we see that $[\mathbf{h}]_{\pi(g)} \subset \underline{\mathbf{g}}$ is $\mathrm{Ad}\pi(g)\mathbf{h}$ and this is the isotropy Lie algebra at $\pi(g)$. Hence we call $[\mathbf{h}]$ the *isotropy bundle.* Henceforth we will denote the projection onto the tangent bundle by $P : \underline{\mathbf{g}} \to [\mathbf{m}]$. Let us observe that for $\xi \in \mathbf{g}$, if we define a vector field $\tilde{\xi}$ by

$$\tilde{\xi}_x = \frac{d}{dt}\bigg|_{t=0} \exp t\xi . x \, , \qquad x \in M,$$

then formula (1) immediately gives us

$$\beta(\tilde{\xi}) = P\xi. \tag{3}$$

As remarked above, the 1-form β is the analogue for reductive homogeneous spaces of the Maurer-Cartan form for Lie groups. We now prove the analogue of the structure equations for β.

Lemma 1.3.

$$d\beta = (1 - \tfrac{1}{2}P)[\beta \wedge \beta].$$

Proof. We begin by differentiating the equivariance relation

$$g^*\beta = \mathrm{Ad}g\beta$$

to obtain

$$L_{\tilde{\xi}}\beta = [\xi, \beta].$$

Applying Cartan's Identity we get

$$i_{\tilde{\xi}}d\beta = [\xi, \beta] - d(i_{\tilde{\xi}}\beta).$$

From (3) we have that

$$i_{\tilde{\xi}}\beta = P\xi$$

so that we obtain from lemma (1.2)

$$d(i_{\tilde{\xi}}\beta) = d(P\xi) = (dP)\xi = [\mathrm{ad}\beta, P]\xi.$$

Thus

$$\begin{aligned} i_{\tilde{\xi}}d\beta &= [\xi, \beta] - [\beta, P\xi] + P[\beta, \xi] \\ &= (1-P)[\xi, \beta] - [\beta, P\xi] \\ &= (1-P)[P\xi, \beta] + [P\xi, \beta] \\ &= (2-P)[P\xi, \beta]. \end{aligned}$$

But

$$\begin{aligned} i_{\tilde{\xi}}[\beta \wedge \beta] &= [\beta(\tilde{\xi}), \beta] - [\beta, \beta(\tilde{\xi})] \\ &= 2[P\xi, \beta], \end{aligned}$$

whence the result follows since the $\tilde{\xi}_x$ span T_xM for all $x \in M$. □

As a simple corollary let us compute the torsion and curvature of the canonical connection:

Corollary 1.4. *Let T, R denote the torsion and curvature respectively of the canonical connection. Then*

$$\beta \circ T = -\tfrac{1}{2}P[\beta \wedge \beta]$$
$$\beta \circ R = -\tfrac{1}{2}[(1-P)[\beta \wedge \beta], \beta].$$

Proof. From proposition (1.1) we have

$$\beta(D_XY) = X\beta(Y) - [\beta(X), \beta(Y)]$$

since we are in the adjoint representation. Thus

$$\begin{aligned} \beta(T(X,Y)) &= \beta(D_XY - D_YX - [X,Y]) \\ &= d\beta(X,Y) - 2[\beta(X), \beta(Y)], \end{aligned}$$

i.e.

$$\beta \circ T = d\beta - [\beta \wedge \beta]$$

and the formula for the torsion follows immediately from lemma (1.3).

As for the curvature,

$$\begin{aligned} \beta(R(X,Y)Z) &= \beta(D_XD_YZ - D_YD_XZ - D_{[X,Y]}) \\ &= [d_X - \mathrm{ad}\beta(X), d_Y - \mathrm{ad}\beta(Y)]\beta(Z) - [X,Y]\beta(Z) + [\beta[X,Y], \beta(Z)] \end{aligned}$$

$$= -\mathrm{ad}d\beta(X,Y)\,\beta(Z) + \mathrm{ad}[\beta(X),\beta(Y)]\,\beta(Z),$$

whence

$$\beta\circ R = \tfrac{1}{2}\mathrm{ad}[\beta\wedge\beta]\circ\beta - \mathrm{ad}d\beta\circ\beta$$

and the result now follows from lemma (1.3). □

Remark. It is clear from Corollary (1.4) that the canonical connection is torsion-free if and only if $[\mathbf{m},\mathbf{m}]\subset\mathbf{h}$, that is, if and only if

$$\mathbf{g} = \mathbf{h} + \mathbf{m}$$

is a symmetric decomposition of $\mathbf{g}$. Further, if M has an invariant metric, the above condition is precisely that the canonical and Levi-Civita connections coincide. Thus these connections coincide for symmetric spaces. Returning now to the general setting of lemma (1.2), we observe that flat differentiation followed by the projection P_i defines a connection in $[W_i]$. It is natural to ask when either of these connections coincides with the canonical one.

Lemma 1.5. *Let* $\underline{V}=[W_1]+[W_2]$ *be a G-invariant splitting and let* D_1 *be the connection in* $[W_1]$ *given by*

$$D_1 = P_1\circ d.$$

Then

$$D_1 = d - P_2\sigma(\beta)\ .$$

Further, $D_1 = D$ *if and only if* $\mathbf{m}.W_1\subset W_2$ *and in this case* $\sigma(\beta)\,|\,[W_1]$ *is the second fundamental form of* $[W_1]$ *in* $\underline{V}$.

Proof. We have from lemma (1.2),

$$\begin{aligned} D_1 = P_1\circ d = d - dP_1 &= d - [\sigma(\beta),P_1] \\ &= d - \sigma(\beta) + P_1\sigma(\beta) \\ &= d - P_2\sigma(\beta). \end{aligned}$$

Thus $D_1 = D$ if and only if

$$\sigma(\beta) = P_2\sigma(\beta)$$

on $[W_1]$, if and only if

$$P_1\sigma(\beta)P_1 = 0,$$

or, by equivariance, if and only if

$$\mathbf{m}.W_1\subset W_2.$$

Lastly, we note that the second fundamental form of $[W_i]$ in $\underline{V}$ is given by

$$P_2\circ d\,|\,C^\infty([W_1]) = P_2\circ\sigma(\beta)\,|\,C^\infty([W_1])$$

and the lemma follows. □

Remark. Applying this to $\underline{\mathbf{g}} = [\mathbf{h}] + [\mathbf{m}]$ we see that the canonical connection on $TM = [\mathbf{m}]$ coincides with $P\circ d$ if and only if $[\mathbf{m},\mathbf{m}]\subset\mathbf{h}$. Thus for Riemannian symmetric spaces all three connections we have considered coincide. This will be of importance when we come to study maps into symmetric spaces, for it gives us two simple formulas for the Levi-Civita connection, as either the canonical connection, or the projection of flat differentiation in the ambient trivial bundle. An important special case is the Lie group G itself which we consider next as an example.

Example The group G may be viewed as a symmetric space for $G\times G$ with action

$$(g_1,g_2)g = g_1 g g_2^{-1}$$

so that the isotropy subgroup at e is the diagonal subgroup $H = \{(g,g): g\in G\}$ and the involution on $G\times G$ is given by

$$\tau_0(g_1,g_2) = (g_2,g_1)$$

Thus $H = (G\times G)^{\tau_0}$ and

$$\mathbf{m} = (\mathbf{g}+\mathbf{g})^{-\tau_0} = \{(\xi,-\xi): \xi\in\mathbf{g}\} \equiv \mathbf{g}$$

To calculate β, take $g\in G$, $X\in T_gG$ and then we have

$$\begin{aligned}\beta_g(X) &= \beta_g((g,e)_*(g_{-1},e)_*X) \\ &= ((g,e)^*\beta)_g(L_{g^{-1}*}X) \\ &= \mathrm{Ad}(g,e)\,(\beta_e(L_{g^{-1}*}X)) \\ &= \mathrm{Ad}(g,e)\,(\tfrac{1}{2}L_{g^{-1}*}X,-\tfrac{1}{2}L_{g^{-1}*}X) \\ &= (\tfrac{1}{2}R_{g^{-1}*}X,-\tfrac{1}{2}L_{g^{-1}*}X).\end{aligned}$$

Thus if θ^L, θ^R are the right- and left-invariant Maurer-Cartan forms,

$$\beta = (\tfrac{1}{2}\theta^R,-\tfrac{1}{2}\theta^L).$$

Substituting this formula into that of proposition (1.1) and projecting onto the second factor gives us

$$\theta^L(D_XY) = X(\theta^L(Y)) + \tfrac{1}{2}[\theta^L(X),\theta^L(Y)]$$

so that identifying TG with $\underline{\mathbf{g}}$ via θ^L we see that

$$D = d + \tfrac{1}{2}\mathrm{ad}\theta^L.$$

Thus our canonical connection is the $\frac{1}{2}$-connection of Cartan-Schouten.

Returning to general reductive homogeneous spaces M, if $\varphi: N\to M$ is a smooth map of a manifold N into our homogeneous space, it is easy to see that our constructions are functorial. Indeed, the (straightforward) proof of the following lemma is left to the reader.

Lemma 1.6. *Let $\sigma: G\to \mathrm{End}(V)$ be a representation, $\underline{V}$ the trivial bundle and $\varphi: N\to M$ a smooth map. Then $\varphi^{-1}\underline{V}$ is trivial and the pull-back of the canonical connection on $\underline{V}$ (over N) is given by*

$$\varphi^{-1}D = d - \sigma(\varphi^*\beta).$$

As an application of our methods let us now prove the well known result that a symmetric space G/K may be totally geodesically immersed in G (c.f. [27]). More generally, we compute the condition for the differential of an equivariant map φ of reductive homogeneous spaces to be parallel for the canonical connection.

Proposition 1.7. *Let N_i, $i=1,2$, be reductive homogeneous G_i-spaces with reductive summands $\mathbf{m}_i$ and $\varphi: N_1 \to N_2$ be a map equivariant with respect to a homomorphism $\rho: G_1 \to G_2$ so*

$$\varphi(gx) = \rho(g)\varphi(x)$$

for $x \in N_1$, $g \in G_1$. Then $d\varphi$ is parallel with respect to the canonical connections if and only if, at some $x \in N_1$,

$$[(1-\varphi^{-1}P_2)\rho([\mathbf{m}_1]_x), \varphi^{-1}P_2\rho([\mathbf{m}_1]_x)] \equiv 0.$$

Here $P_2: \underline{\mathbf{g}} \to [\mathbf{m}_2]$ is projection along the isotropy bundle of N_2.

Remark. This condition is satisfied if $\rho([\mathbf{m}_1]_x) \subset [\mathbf{m}_2]_{\varphi(x)}$.

We begin the proof with a lemma of independent interest:

Lemma 1.8. *Under the hypotheses of (1.7), let β_i denote the Maurer-Cartan form of N_i. Then*

$$\varphi^*\beta_2 = \varphi^{-1}P_2 \circ \rho \circ \beta_1 \ .$$

Proof. Let $X \in T_x N_1$ then

$$X = \frac{d}{dt}\bigg|_{t=0} \exp t\beta_1(X).x,$$

so

$$\begin{aligned} d\varphi(X) &= \frac{d}{dt}\bigg|_{t=0} \varphi(\exp t\beta_1(X).x) \\ &= \frac{d}{dt}\bigg|_{t=0} \rho(\exp t\beta_1(X)).\varphi(x) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp t\rho(\beta_1(X)).\varphi(x) \end{aligned}$$

whence

$$\varphi^*\beta_2(X) = (\varphi^{-1}P_2)\rho\beta_1(X).$$

□

Proof of (1.7). If D^1, D^2 are the canonical connections we must show that

$$Dd\varphi(X,Y) = \varphi^{-1}D_X^2 d\varphi(Y) - d\varphi(D_X^1 Y) \equiv 0.$$

Using (1.8), we have

$$\begin{aligned} \varphi^{-1}\beta_2(Dd\varphi(X,Y)) &= \varphi^{-1}D_X^2(\varphi^{-1}P_2\rho\beta_1(Y)) - \varphi^{-1}P_2\rho\beta_1(D_X^1 Y) \\ &= \varphi^{-1}P_2\{\varphi^{-1}D_X^2(\rho\beta_1(Y)) - \rho\beta_1(D_X^1 Y)\} \end{aligned}$$

since $D^2 P_2 \equiv 0$, so

$$\begin{aligned} \varphi^{-1}\beta_2(Dd\varphi(X,Y)) &= -\varphi^{-1}P_2\{[\varphi^*\beta_2(X), \rho\beta_1(Y)] - \rho[\beta_1(X), \beta_1(Y)]\} \\ &= -\varphi^{-1}P_2[(\varphi^{-1}P_2 - 1)\rho\beta_1(X), \rho\beta_1(Y)] \\ &= [(1-\varphi^{-1}P_2)\rho\beta_1(X), \varphi^{-1}P_2\rho\beta_1(Y)]. \end{aligned}$$

The proposition now follows by evaluating the above formula at any point $x \in N_1$, which is sufficient by equivariance. □

Corollary 1.9. *Let G/K be a symmetric space with involution τ_0 at eK. Then we have a totally geodesic immersion $\varphi: G/K \to G$ defined by*

$$\varphi(gK) = g^{\tau_0} g^{-1}.$$

Proof. Define $\rho: G \to G \times G$ by $\rho(g) = (g^{\tau_0}, g)$. Then φ is equivariant with respect to ρ and for $\xi \in \mathbf{m}$

$$\rho(\xi) = (\tau_0 \xi, \xi) = (-\xi, \xi)$$

whence ρ preserves the symmetric decompositions of G/K and $G = G \times G/G$ and so φ is totally geodesic by lemma (1.7). □

Let us conclude this chapter with an example to illustrate the concepts we have introduced:

Let $G_{r,n}$ denote the complex Grassmannian of r-planes in $\mathbf{C}^n$. Clearly, the unitary group $\mathbf{U}(n)$ acts transitively on $G_{r,n}$ so that $G_{r,n}$ is a homogeneous space. Let us take as a base point V_0, the span of the first r elements of the canonical basis of $\mathbf{C}^n$. Then the stabiliser H of V_0 is isomorphic to $\mathbf{U}(r) \times \mathbf{U}(n-r)$. Further, letting τ_0 denote the involutive automorphism of $\mathbf{U}(n)$ obtained by conjugating with

$$J_r = \begin{cases} 1 & \text{on } V_0 \\ -1 & \text{on } V_0^\perp \end{cases}$$

we see that $H = \mathbf{U}(n)^{\tau_0}$ so that $G_{r,n}$ is a symmetric space. The Lie algebra $\mathbf{u}(n)$ of $\mathbf{U}(n)$ is the algebra of $n \times n$ skew-Hermitian matrices and the corresponding symmetric decomposition is given by

$$\mathbf{u}(n) = \mathbf{h} \oplus \mathbf{m}$$

where

$$\mathbf{h} = \{A \in \mathbf{u}(n) \colon AV_0 \subset V_0\}$$

$$\mathbf{m} = \{A \in \mathbf{u}(n) \colon AV_0 \subset V_0^\perp, \quad AV_0^\perp \subset V_0\}$$

Now, we have

$$[V_0]_{gV_0} = gV_0 \quad \text{for } g \in \mathbf{U}(n),$$

so that $[V_0] \subset \underline{\mathbf{C}}^n$ is the tautological bundle $T \to G_{r,n}$ whose fibre at $W \in G_{r,n}$ is W itself. Similarly,

$$[\mathbf{m}]_W = \{A \in \mathbf{u}(n) \colon AW \subset W^\perp, \quad AW^\perp \subset W\}.$$

Thus, the trivial bundle $\underline{\mathbf{C}}^n$ admits a $\mathbf{U}(n)$-invariant splitting as the direct sum of the tautological bundle and its orthocomplement. Moreover, by lemma (1.5), the Maurer-Cartan form of $G_{r,n}$ is the sum of the second fundamental forms of T and $T^\perp$. This enables us to reduce many questions concerning the geometry of $G_{r,n}$ to the study of sub-bundles of $\underline{\mathbf{C}}^n$ and their second fundamental forms: an approach which recently proved to be useful in the study of harmonic maps into Grassmannians (c.f. [22, 21]).

Finally, an easy computation shows that the totally geodesic embedding of $G_{r,n}$ into $\mathbf{U}(n)$ provided by lemma (1.9) is just $J_r(1-2P)$. So, after a left translation by J_r, we see that the family of Grassmannians $G_{r,n}$, $r=0,\ldots,n$ are realised totally geodesically in $\mathbf{U}(n)$ as the components of the set

$$\{g\in\mathbf{U}(n):\ g^2=1\}$$

as was observed by Uhlenbeck [72].

Chapter 2. Harmonic maps and twistor spaces

Let (M,g) and (N,h) be Riemannian manifolds and $\varphi:M \to N$ a smooth map. The *energy* of φ, denoted $E(\varphi)$, is given by

$$E(\varphi) = \tfrac{1}{2}\int_M \operatorname{trace}_g \varphi^* h \, dvol_M .$$

A map is *harmonic* if it extremizes the energy with respect to all compactly supported variations. The associated Euler-Lagrange equation is

$$\tau_\varphi \equiv \operatorname{trace}_g \nabla d\varphi = 0$$

where ∇ is the connection on $T^*M \otimes \varphi^{-1}TN$ induced by the Levi-Civita connections of M and N. The quantity $\tau_\varphi \in C^\infty(\varphi^{-1}TN)$ is called the *tension field.*

For surveys of harmonic map theory see [31, 32].

In the case the domain is two-dimensional, there are a number of special features to the theory: the energy is conformally invariant for the domain metric, and the tension field has a particularly simple form. Indeed, if $z=x+iy$ is a local complex coordinate on M then the tension field is given, up to a conformal factor, by

$$(\varphi^{-1}\nabla^N)_{\frac{\partial}{\partial \bar z}} \varphi_*(\frac{\partial}{\partial z}).$$

This is in fact a Cauchy-Riemann equation as the following theorem shows:

Theorem 2.1. [48] *Let $E \to M$ be a complex vector bundle over a Riemann surface M with connection ∇. Then there is a unique holomorphic structure on E compatible with ∇, that is: a local section σ of E is holomorphic if and only if $\nabla_{\bar Z}\sigma = 0$ for all (1,0) vectors Z.*

We call this the *Koszul-Malgrange* holomorphic structure induced by ∇.

Writing $d\varphi = dz \otimes \delta + d\bar z \otimes \bar\delta$ we obtain the well-known

Proposition 2.2. *Let M be a Riemann surface and $\varphi:M \to N$ a map. Then φ is harmonic if and only if $dz \otimes \delta$ is a holomorphic section of $\kappa_M \otimes \varphi^{-1}TN^{\mathbb{C}}$.*

Here κ_M denotes the canonical line bundle of M. Since non-trivial holomorphic sections have only isolated zeros we can make the

Definition. If $\varphi:M \to N$ is a harmonic map of a Riemann surface, the *ramification index* of φ, denoted by r_φ, is the number of zeros of $dz \otimes \delta$ counted with their multiplicities.

Remark. If φ is not constant, r_φ is finite and non-negative. In this case we denote by L_φ the line subbundle of $\varphi^{-1}TN^{\mathbb{C}}$ spanned locally by δ where the latter is non-zero. It follows that $dz \otimes \delta$ is a non-zero section of the line bundle $\kappa_M \otimes L_\varphi$ and hence that $r_\varphi = c_1(\kappa_M \otimes L_\varphi)[M]$.

Following work of Calabi [23], Eells-Wood [34] and more recently Eells-Salamon [33] and Rawnsley [61], it is natural to attempt to study harmonic maps of surfaces by relating them to holomorphic maps (which may have values in an associated almost complex manifold). With this in mind we make the following definition:

Definition. A twistor fibration $\pi:Z \to N$ (with twistor space Z) is a fibration of an almost complex manifold Z over a Riemannian manifold N with the following property:

If M is almost Hermitian with co-closed Kähler form and $\psi:M \to Z$ is holomorphic then $\pi \circ \psi: M \to N$ is harmonic.

Remark. The above definition of a twistor space is chosen purely for its relevance to harmonic map theory and differs from others in the literature (for instance we do not demand that the fibres be complex submanifolds of Z).

Example [33]. Let N be a $2n$-dimensional Riemannian manifold and let $\pi:J(N) \to N$ be the bundle of Hermitian almost complex structures on N. Thus

$$J_x(N) = \{J \in \mathrm{End}(T_xN) : J^2 = -1,\ J \text{ skew-symmetric}\}.$$

This bundle is associated to the orthonormal frame bundle of N with typical fibre $O(2n)/U(n)$ which is a Hermitian symmetric space. Thus the vertical distribution $V = \ker \pi_*$ inherits an almost complex structure J^V. Further the Levi-Civita connection on N induces a splitting

$$TJ(N) = V \oplus H$$

where $H \cong \pi^{-1}TN$ and thus acquires a tautological almost complex structure J^H given by

$$J_j^H = j.$$

This gives two almost complex structures on $J(N)$: $J_1 = J^V + J^H$ and $J_2 = (-J^V) + J^H$. This second almost complex structure J_2 makes $\pi:J(N) \to N$ into a twistor fibration. To see this, first observe that a map $\psi:M \to J(N)$ with $\pi \circ \psi = \varphi$ is the same as an almost complex structure on $\varphi^{-1}TN$ or, equivalently, a maximally isotropic subbundle $\underline{\psi}$ (the (1,0) vectors) of $\varphi^{-1}TN^{\mathbb{C}}$. We have

Proposition 2.3. [61] *Let $\psi:M \to J(N)$ be a map of an almost Hermitian manifold and set $\varphi = \pi \circ \psi$. Then ψ is holomorphic with respect to J_1 if and only if*

$(i)_1 \quad \varphi^{-1}\nabla_{\bar{Z}}^N C^\infty(\underline{\psi}) \subset C^\infty(\underline{\psi}), \quad Z \in C^\infty(T^{1,0}M)$;

$(ii) \quad \varphi_*(T^{1,0}M) \subset \underline{\psi}$,

and holomorphic with respect to J_2 if and only if

$(i)_2 \quad \varphi^{-1}\nabla_{Z}^N C^\infty(\underline{\psi}) \subset C^\infty(\underline{\psi}), \quad Z \in C^\infty(T^{1,0}M)$;

$(ii) \quad \varphi_*(T^{1,0}M) \subset \underline{\psi}$.

Remarks. (a) Conditions (i) and (ii) correspond with the holomorphicity of the vertical and horizontal parts of the differential of ψ respectively.

(b) If $\dim M = 2$ we see from (2.1) that $(i)_2$ is equivalent to $\underline{\psi}$ being a holomorphic subbundle of $\varphi^{-1}TN^{\mathbb{C}}$, while (ii) is equivalent to $L_\varphi \subset \underline{\psi}$.

Now we have

Theorem 2.4. [33] $\pi:(J(N),J_2)\to N$ *is a twistor fibration.*

Proof. Let M be an almost Hermitian manifold with co-closed Kähler form and let $Z_1,\ldots,Z_m$ be a local orthonormal frame field for $T^{1,0}M$. Then the tension field of φ is given by

$$\tau_\varphi = \sum \nabla d\varphi(\bar{Z}_i,Z_i) = \sum \varphi^{-1}\nabla^N_{\bar{Z}_i}\varphi_* Z_i - \sum \varphi_* \nabla^M_{\bar{Z}_i} Z_i .$$

The condition on the Kähler form ensures that $\sum \nabla^M_{\bar{Z}_i} Z_i \in T^{1,0}M$ so that $(i)_2$ and (ii) of proposition (2.3) imply that both summands of τ_φ are contained in $\underline{\psi}$. Thus τ_φ is both real and isotropic and hence zero. □

Clearly it is desirable to know which harmonic maps are projections of J_2-holomorphic maps. For this we restrict attention to Riemann surfaces as domains. Since $\varphi^{-1}TN$ has an almost complex structure coming from such a ψ, it is necessarily an oriented bundle and so $w_1(\varphi^{-1}TN)=0$. This gives a topological restriction on φ, namely $\varphi^* w_1(N)=0$. There is a geometrical restriction also, for (ii) of proposition (2.3) implies that φ is (weakly) conformal. In fact these are the only restrictions, a result which is implicit in theorem 9.10 of [61]. For completeness, we give a proof here:

Theorem 2.5. *A map* $\varphi:M\to N^{2n}$ *of a Riemann surface has a* J_2*-holomorphic lift* $\psi:M\to J(N)$ *if and only if it is weakly conformal, harmonic and* $\varphi^* w_1(N)=0$.

Proof. We have already seen that projections of J_2-holomorphic maps are weakly conformal harmonic with $\varphi^* w_1(N)=0$.

For the converse, let φ be weakly conformal and harmonic and L_φ the holomorphic line bundle locally spanned by δ. Since φ is weakly conformal, L_φ is isotropic. We must extend L_φ to a maximally isotropic holomorphic sub-bundle $\underline{\psi}$ of $\varphi^{-1}TN^{\mathbb{C}}$ and then the corresponding map $\psi:M\to J(N)$ will be our desired lift by (2.3).

To extend an isotropic holomorphic sub-bundle V of rank k to one of rank $k+1$ amounts to finding an isotropic holomorphic line sub-bundle of $E=V^\circ/V$, where V° is the polar of V. For this, it suffices to find an isotropic meromorphic section of E. Such a section is a solution of a quadratic equation in rank E unknowns over the function field of meromorphic functions on M. According to Lang [49], such a solution exists if rank $E\geq 3$, i.e. if $2n-2k\geq 3$. Thus, starting with L_φ, we can find successive extensions until we have a rank $(n-1)$ isotropic holomorphic bundle V containing L_φ.

Now V°/V has rank 2 and will admit an isotropic line sub-bundle if and only if it is orientable. This, in turn, happens precisely when $\varphi^{-1}TN$ is orientable which is equivalent to the vanishing of $\varphi^* w_1(N)=0$. Lastly, it is easy to check that a line sub-bundle of a rank 2 holomorphic bundle that is isotropic with respect to a holomorphic non-degenerate bilinear form is itself holomorphic. The theorem now follows. □

This result is an abstract existence theorem and does not give a means to construct a J_2-holomorphic lift explicitly. In order to find a more practical means for constructing lifts (which we do by finding holomorphic maximal isotropic subbundles of $\varphi^{-1}TN^{\mathbb{C}}$) we have recourse to the Harder-Narasimhan filtration of a holomorphic vector bundle on a Riemann surface which