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ANNALS OF DISCRETE MATHEMATICS

17

Combinatorial Mathematics

C. BERGE
D. BRESSON
P. CAMION
J.F. MAURRAS
F. STERBOUL
editors

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COMBINATORIAL MATHEMATICS

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COMBINATORIAL MATHEMATICS

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PREFACE

The International Colloquium on Graph Theory and Combinatorics was held in Marseille from June 14 to June 19, 1981, under the auspices of the National Centre for Scientific Research (C.N.R.S.). The Colloquium — the second of its kind organized by the C.N.R.S. — was a great success. In addition to the invited talks, 95 participants from many parts of the world were invited to present communications in combinatorics: graph theory, hypergraphs, designs, coding. This volume contains most of the papers that were presented: new results or surveys.

We wish to thank the organizations that have contributed to the success of the Colloquium:

Direction des Recherches, Etudes et Techniques, Ministère des Armées (D.R.E.T.),

I.B.M. France,

Mairie de Marseille,

Société Mathématique de France,

International Mathematical Union,

Université de Paris XII.

The lectures were given at the University of Marseille–Luminy, and the participants had the pleasure of being the first guests of the Centre International de Rencontres Mathématiques (C.I.R.M.). We wish to express our gratitude to this new centre and particularly to Mrs Anne Litman for her many contributions to making this colloquium a success. We are also indebted to Mrs Martine Ulrich for her thoughtful help at Luminy as well as for her careful preparation of the final manuscript. In addition, it is a pleasure to acknowledge the continuing support of the Ecole des Hautes Etudes en Sciences Sociales (C.M.S.) that helped make possible the organization of the colloquium.

Last but not least, we wish to thank the North-Holland Publishing Company for the smooth and efficient job they did in preparing this volume.

THE EDITORS

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EXTREMAL BASES OF ADDITIVE PERMUTATIONS

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Let $X = (x_1, x_2, \dots, x_k)$ be a vector with relatively prime integer components $x_1 < x_2 < \cdots < x_k$, $k \geq 1$. Then X is called a basis of additive permutations (A-basis) if there exists a vector $Y = (y_1, y_2, \dots, y_k)$ such that both $(y_1, y_2, \dots, y_k)$ and $(x_1 + y_1, x_2 + y_2, \dots, x_k + y_k)$ are permutations of X. We will write $X + Y$ for $(x_1 + y_1, \dots, x_k + y_k)$. The concept of an A-basis was introduced by Kotzig [7]. All A-bases of cardinalities five and less are described in [6] and in the appendix of [3]; all A-bases of cardinality six together with their additive permutations have been found in [1]. Additional results on additive permutations have been presented, e.g., in [4], [5], [8], [9], [10].

If X is an A-basis, then the following symbols will be used to denote certain parameters of X:

$|X|$ — the cardinality (number of components) of X;
$\pi(X)$ — the number of positive components of X;
$\nu(X)$ — the number of negative components of X;
$\omega(X)$ — the number of odd components of X;
$\psi(X)$ — the number of additive permutations of X.

The following three theorems give some extremal values of the above parameters.

Theorem A. *If* $|X| > 3$ *then* $\pi(X) \leq |X| - 3$, $\nu(X) \leq |X| - 3$.

Proof. For the proof, see [1, Theorem 1].

Theorem B. *If* $|X| \geq 3$ *then* $\omega(X) \equiv 0 \pmod 2$ *and* $2 \leq \omega(X) \leq \frac{2}{3}|X|$.

Proof. For the proof, see [7, Theorem 3].

Theorem C. *If* $|X| \geqslant 3$ *then* $\psi(X) \equiv 0 \pmod 2$, *hence* $\psi(X) \geqslant 2$.

Proof. Let Y be an additive permutation of X, and let Y' be the inverse permutation to Y; then Y' is clearly an additive permutation to X. To complete the proof it is enough to show that $Y \neq Y'$. Since Y cannot be an identical permutation there exists an x in X such that the corresponding element y in Y is different from x. We have $y \in X$, and if $Y' = Y$, x is the element corresponding to y in Y'. Then $X + Y$ contains $x + y$ in two positions, so $X + Y$ is not a permutation of X.

On the basis of the above theorems, we may want to investigate the following extremal bases of additive permutations:

(a) A-bases with the minimum/maximum number of positive (negative) components.

(b) A-bases with the minimum/maximum number of odd (even) components.

(c) A-bases with the minimum number of additive permutations.

(Very little is known about the maximum number of additive permutations; however, the results presented in [4] imply that, for $X_r = (-r, -r+1, \ldots, r-1, r)$, $\psi(X_r) \to +\infty$ as $r \to +\infty$ if $r \equiv 0$ or $1 \pmod 4$.)

Let us now summarize what is known about these classes of extremal A-bases.

(a) All A-bases of cardinality at least five with exactly two negative elements have been described by the authors in [3] and are listed in Table 1 (which also comes from [3]; the cases listed refer to [3] as well). The reader will observe that, in Table 1, $|X| = n + 3$ with $n \geqslant 2$ and the subscripts -2, -1, 0 corresponding to the two negative components and to this zero component of X respectively. Some preliminary attempts at finding all A-bases with $|X| - 3$ positive and three negative components have indicated that the number of such A-bases will grow very rapidly with the increase in the cardinality.

(b) A-bases with the given number of odd elements have been studied in [2] where the following theorem has been proved: Let k be an integer, $k \geqslant 3$, $k \neq 4$. Let m be an even number such that $2 \leqslant m \leqslant \frac{2}{3}k$. Then there exists an A-basis X such that $|X| = k$, $\omega(X) = m$.

This theorem implies that for any $k \geqslant 3$, $k \neq 4$, there exists an A-basis of cardinality k with the minimum (maximum) number of odd/even components. The reader will observe that the A-bases listed in Table 1, which do not depend on the parameters r, s, contain exactly two odd components. For the remaining A-bases listed, r, s can be chosen in such a way that each A-basis obtained contains exactly two odd components.

(c) According to Theorem C, $\psi(X) \geqslant 2$ if $|X| \geqslant 3$. For any A-basis of cardinality six or more, listed in Table 1, we have $\psi(X) = 2$. For $X = (-2, -1, 0, 1, 2)$ (the only A-basis of cardinality five) Table 1 yields four additive permutations and two more can be found in [3]. Theorem D below shows that there exists a class of A-bases for which $\psi(X) \geqslant 4$. For this theorem, we will need the following definition.

Definition. An A-basis $X = (x_1, \dots, x_k)$ is called *symmetric* if $x_i = -x_{k+1-i}$ for $i = 1, 2, \dots, k$.

Theorem D. *Let X be a symmetric A-basis, $|X| \geqslant 3$. Then $\psi(X) \geqslant 4$.*

Table 1

Case \ j		-2	-1	0	1	2	3	4	$\cdots$	$n-2$	$n-1$	n
1	X	-2^{n-1}	-2^{n-2}	0	1	2	2^2	2^3	$\cdots$	2^{n-3}	2^{n-2}	$1+2^{n-2}$
$(n \geqslant 2)$	Y	2^{n-2}	$1+2^{n-2}$	-2^{n-1}	1	2	2^2	2^3	$\cdots$	2^{n-3}	-2^{n-2}	0
	Z	-2^{n-2}	1	-2^{n-1}	2	2^2	2^3	2^4	$\cdots$	2^{n-2}	0	$1+2^{n-2}$
	Y'	0	2^{n-2}	$1+2^{n-2}$	1	2	2^2	2^3	$\cdots$	2^{n-3}	-2^{n-1}	-2^{n-2}
	Z'	-2^{n-1}	0	$1+2^{n-2}$	2	2^2	2^3	2^4	$\cdots$	2^{n-2}	-2^{n-2}	1
2.1.	X	-2^{n-1}	$1-2^{n-1}$	0	1	2	2^2	2^3	$\cdots$	2^{n-3}	2^{n-2}	2^{n-1}
$(n \geqslant 2)$	Y	2^{n-1}	0	-2^{n-1}	1	2	2^2	2^3	$\cdots$	2^{n-3}	2^{n-2}	$1-2^{n-1}$
	Z	0	$1-2^{n-1}$	-2^{n-1}	2	2^2	2^3	2^4	$\cdots$	2^{n-2}	2^{n-1}	1
	Y'	0	2^{n-1}	$1-2^{n-1}$	1	2	2^2	2^3	$\cdots$	2^{n-3}	2^{n-2}	-2^{n-1}
	Z'	-2^{n-1}	1	$1-2^{n-1}$	2	2^2	2^3	2^4	$\cdots$	2^{n-2}	2^{n-1}	0
2.2.1.	X	$-a_{n-3}-s$	$-a_{n-4}$	0	r	s	a_0	a_1	$\cdots$	a_{n-5}	a_{n-4}	$a_{n-4}+s$
$(n \geqslant 4)$	Y	$a_{n-4}+s$	a_{n-4}	$-a_{n-3}-s$	0	r	a_0	a_1	$\cdots$	a_{n-5}	s	$-a_{n-4}$
	Z	$-a_{n-4}$	0	$-a_{n-3}-s$	r	a_0	a_1	a_2	$\cdots$	a_{n-4}	$a_{n-4}+s$	s
	Y'	0	$a_{n-4}+s$	r	s	a_{n-4}	a_0	a_1	$\cdots$	a_{n-5}	$-a_{n-4}$	$-a_{n-3}-s$
	Z'	$-a_{n-3}-s$	s	r	a_0	$a_{n-4}+s$	a_1	a_2	$\cdots$	a_{n-4}	0	$-a_{n-4}$
2.2.2.1.	X	$1-2^{n-1}$	$1-2^{n-2}$	0	1	2	2^2	2^3	$\cdots$	2^{n-3}	$2^{n-2}-1$	2^{n-2}
$(n \geqslant 4)$	Y	2^{n-2}	$2^{n-2}-1$	$1-2^{n-1}$	1	2	2^2	2^3	$\cdots$	2^{n-3}	0	$1-2^{n-2}$
	Z	$1-2^{n-2}$	0	$1-2^{n-1}$	2	2^2	2^3	2^4	$\cdots$	2^{n-2}	$2^{n-2}-1$	1
	Y'	0	2^{n-2}	$2^{n-2}-1$	1	2	2^2	2^3	$\cdots$	2^{n-3}	$1-2^{n-2}$	$1-2^{n-1}$
	Z'	$1-2^{n-1}$	1	$2^{n-2}-1$	2	2^2	2^3	2^4	$\cdots$	2^{n-2}	0	$1-2^{n-2}$
2.2.2.2.	X	$-a_{n-3}-r$	$-a_{n-4}$	0	r	s	a_0	a_1	$\cdots$	a_{n-5}	a_{n-4}	$a_{n-4}+r$
$(n \geqslant 4)$	Y	$a_{n-4}+r$	a_{n-4}	$-a_{n-3}-r$	s	0	a_0	a_1	$\cdots$	a_{n-5}	r	$-a_{n-4}$
	Z	$-a_{n-4}$	0	$-a_{n-3}-r$	a_0	s	a_1	a_2	$\cdots$	a_{n-4}	$a_{n-4}+r$	r
	Y'	0	$a_{n-4}+r$	s	a_{n-4}	r	a_0	a_1	$\cdots$	a_{n-5}	$-a_{n-4}$	$-a_{n-3}-r$
	Z'	$-a_{n-3}-r$	r	s	$a_{n-4}+r$	a_0	a_1	a_2	$\cdots$	a_{n-4}	0	$-a_{n-4}$

Please note:

(1) Only A-bases with cardinality five or more and exactly two negative elements are listed.

(2) If Y is an additive permutation of X, and $Z = X + Y$, then Y' is the inverse permutation, and $Z' = X + Y'$.

(3) If r, s are used, they denote relatively prime integers such that $0 < r < s$. In that case, a_k denotes $2^k(r+s)$.

Proof. Let Y be an additive permutation of X. Three cases (in general, not mutually exclusive) will be considered. The term 'monocycle' will refer to a cycle of length one.

(1) Let Y have at least two cycles of length two or more. Let $\bar{Y}$ be obtained from Y by changing the orientation in one of the cycles. Then Y, $\bar{Y}$ and their inverse permutations form a set of four different additive permutations of X.

(2) Let Y consist of one cycle, containing all nonzero components of X; if 0 is in X, then 0 may either be in the above cycle, or it may form a monocycle. (In the latter case, we have $|X| > 5$.) Both cases being similar, only the case of a unique cycle will be considered. Let $Z = X + Y$. The symmetry of X together with the relation $X + (-Z) = -Y$ imply that $-Z$ also is an additive permutation of X. We will show that Y, $-Z$, and their respective inverse permutations Y', $-Z'$ form a set of four different additive permutations of X. Since $Y' \neq Y$, $-Z' \neq -Z$ (see the proof of Theorem C) it is sufficient to show that $Y' \neq -Z$.

Let us write now $X = (x_1, x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ where $x_1 x_{i_1} x_{i_2} \cdots x_{i_k}$ is the cycle of Y. (The reader will observe that — to simplify the notation — the components of X do not appear in their natural order.) Then

$$Y = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_1),$$

$$Y' = (x_{i_k}, x_1, \ldots, x_{i_{k-2}}, x_{i_{k-1}}),$$

$$Z = X + Y = (x_1 + x_{i_1}, x_{i_1} + x_{i_2}, \ldots, x_{i_{k-1}} + x_{i_k}, x_{i_k} + x_1),$$

$$\tilde{Z} = X + Y' = (x_1 + x_{i_k}, x_{i_1} + x_1, \ldots, x_{i_{k-1}} + x_{i_{k-2}}, x_{i_k} + x_{i_{k-1}}).$$

If $\psi(X) = 2$ we have necessarily $Z = -Y'$, $\tilde{Z} = -Y$; taking the last two components into consideration we get $x_{i_{k-1}} + x_{i_k} = -x_{i_{k-2}}$ and $x_{i_k} + x_{i_{k-1}} = -x_1$ which implies $x_{i_{k-2}} = x_1$, and this contradicts our assumption.

(3) Let Y have at least one monocycle (x_i) where $x_i \neq 0$. Let Y be an additive permutation of X, $Z = X + Y$. If $\psi(X) = 2$ then $Y' = -Z$. If y_i, y'_i, z_i denote the ith component of Y, Y', Z respectively, then $y'_i = x_i$ (from the definition of the inverse permutation), and $y'_i = -2x_i$ (from the relation $Y' = -Z$) and this is impossible if $x_i \neq 0$.

The statement of Theorem D is stronger than it may seem at first glance: The equality $\psi(X) = 4$ is satisfied for the symmetric A-basis of cardinality 6 presented in [1]. Since no other solution to the equality $\chi(X) = 4$ has been found yet, a stronger statement might be possible if the case $|X| = 6$ is excluded. However, the improvement — if any — will be small because of the following fact: For any odd $k > 3$, a symmetric A-basis X of cardinality k exists such that $\psi(X) = 6$. It is sufficient to choose (with $k = 2r + 1$, $r \geqslant 1$)

$$X = (-2^{r-1}, -2^{r-2}, \ldots, -2, -1, 0, 1, 2, 2^2, \ldots, 2^{r-1}).$$

This A-basis has already been mentioned in the conclusion of [2].

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ORDRE MINIMUM D'UN GRAPHE SIMPLE DE DIAMETRE, DEGRE MINIMUM ET CONNEXITE DONNES

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We give the minimum order of a connected simple graph with given diameter, minimum degree and connectivity.

Notations. Soit G un graphe simple connexe. Pour les définitions des termes suivants voir [1]. On note D le diamètre du graphe G, δ le degré minimum de G et $n(G)$ l'ordre de G. Si k est un entier vérifiant $1 \leqslant k \leqslant \delta$, on note $\mathscr{G}(D, \delta, k)$ l'ensemble des graphes G de connexité $\geqslant k$, de diamètre D et degré minimum δ.

On notera $V(G)$ l'ensemble des sommets de G; si $A \subset V(G)$, on note $|A|$ le cardinal de A.

On cherche à déterminer la fonction $f(D, \delta, k) = \inf\{n(G) \mid G \in \mathscr{G}(D, \delta, k)\}$ et on montre le théorème suivant.

Théorème. (1) $f(1, \delta, k) = \delta + 1$.

(2) $f(2, \delta, k) = \delta + 2$.

(3) *Si* $D \geqslant 3$ *et* $1 \leqslant k \leqslant \delta \leqslant 3k - 1$

$$f(D, \delta, k) = k(D-3) + 2\delta + 2.$$

Si $D \geqslant 3$ *et* $\delta \geqslant 3k - 1$

$$f(D, \delta, k) = (t+1)(\delta+1) + \varepsilon k,$$

où t *et* ε *sont déterminés par*:

$$D = 3t + \varepsilon \qquad t \in \mathbb{N}^* \quad \varepsilon \in \{0, 1, 2\}.$$

Démonstration du théorème. (1) Les graphes de diamètre 1 sont les graphes complets d'où $f(1, \delta, k) = \delta + 1$.

(2) Un graphe G de diamètre 2 n'est pas complet, d'où si son degré minimum est δ, $n(G) \geqslant \delta + 2$. On considère le graphe G_0 égal au graphe complet $K_{\delta+2}$ privé d'une arête (u, v) (Fig. 1). $n(G_0) = \delta + 2$ et $G_0 \in \mathscr{G}(2, \delta, k)$ d'où le résultat.

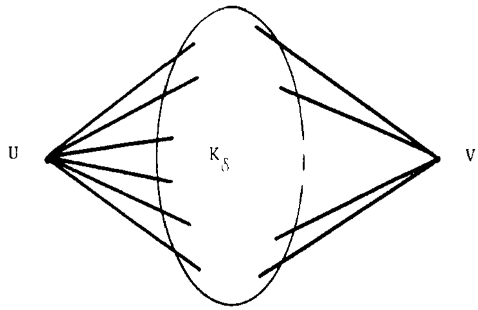

Fig. 1.

(3) Nous supposons dans la suite $D \geqslant 3$.

(a) Kane et Mohanty ont prouvé [2] le théorème suivant:

Si $D \geqslant 3$ *et* $G \in \mathcal{G}(D, \delta, k)$,

$$n(G) \geqslant k(D-3) + 2\delta + 2 = g_1(D, \delta, k).$$

Cette borne n'est pas toujours la meilleure possible.

(b) Nous prouvons ici le lemme.

Lemme. *Si* $D \geqslant 3$ *et* $G \in \mathcal{G}(D, \delta, k)$,

$$n(G) \geqslant (t+1)(\delta+1) + \varepsilon k = g_2(D, \delta, k),$$

où $D = 3t + \varepsilon$ $(t \in \mathbb{N}^*,\ \varepsilon \in \{0, 1, 2\})$

Cette borne n'est pas toujours la meilleure possible et nous montrerons en fait que

$$f(D, \delta, k) = \sup\{g_1(D, \delta, k), g_2(D, \delta, k)\}.$$

Démonstration du lemme. Soit G un graphe dans $\mathcal{G}(D, \delta, k)$ où $D \geqslant 3$. On considère x_0 et x_D deux sommets de G à distance D et $\{x_0, x_1, x_2, \ldots, x_D\}$ un chemin allant de x_0 à x_D. Pour $i \in \{0, \ldots, D\}$ on note $N'(x_i)$ l'ensemble des voisins de x_i n'appartenant pas à l'ensemble $\{x_0, x_1, \ldots, x_D\}$.

Remarques. La distance entre x_0 et x_D étant D:

(1) Deux sommets du chemin ne sont adjacents que s'ils sont consécutifs, donc $N'(x_0)$ et $N'(x_D)$ ont au moins $\delta - 1$ sommets. Pour $1 \leqslant i \leqslant D-1$, $N'(x_i)$ a au moins $\delta - 2$ sommets.

(2) Si $|i-j| \geqslant 3$, $N'(x_i) \cap N'(x_j) = \emptyset$.

Soient $t \in \mathbb{N}^*$ et $\varepsilon \in \{0, 1, 2\}$ définis par $D = 3t + \varepsilon$. En fonction de ε on considère les trois cas suivants:

Premier cas: $D = 3t$

$$n(G) \geqslant |\{x_0, \ldots, x_D\}| + \sum_{i=0}^{t} |N'(x_{3i})|,$$

$$n(G) \geqslant D + 2 + 2(\delta - 1) + (t-1)(\delta - 2),$$

$$n(G) \geqslant (t+1)(\delta + 1).$$

Deuxième cas: $D = 3t + 1$

Soit $S = \{x_0, x_1\} \cup N'(x_0)$ et soit T l'ensemble des sommets de G n'appartenant pas à S. $|S| \geqslant k$, $|T| \geqslant k$, d'où G étant k-connexe, il y a k arêtes disjointes entre S et T. L'une d'elles peut être d'extrémité x_2; la distance entre x_0 et x_D étant égale à D les $k-1$ autres ont pour extrémités dans T des sommets $v_1, v_2, \ldots, v_{k-1}$ n'appartenant ni au chemin $\{x_0, x_1, \ldots, x_D\}$ ni à $N'(x_i)$ pour $i > 3$, donc en particulier à $N'(x_{3i+1})$ pour $i = 1, \ldots, t$

$$n(G) \geqslant |\{x_0, \ldots, x_D\}| + |N'(x_0)| + \sum_{i=1}^{t} |N'(x_{3i+1})| + |\{v_1, \ldots, v_{k-1}\}|,$$

$$n(G) \geqslant 3t + 2 + 2(\delta - 1) + (t-1)(\delta - 2) + k - 1,$$

$$n(G) \geqslant (t+1)(\delta + 1) + k.$$

Troisième cas: $D = 3t + 2$

On minore le nombre de sommets des ensembles disjoints suivants:

$$\{x_0, x_1, \ldots, x_D\}, \quad N'(x_0), \quad N'(x_{3i+1}), \qquad 1 \leqslant i \leqslant t-1,$$

$N'(x_{3t+2})$, V_1 et V_2, où V_1 et V_2 sont définis comme suit:

Soit

$$S_1 = \{x_0, x_1\} \cup N'(x_0); \quad S_2 = \{x_{3t+1}, x_{3t+2}\} \cup N'(x_{3t+2}),$$

$$T_1 = V(G) \backslash S_1; \quad T_2 = V(G) \backslash S_2.$$

V_1 (resp. V_2) est l'ensemble des extrémités autres que x_2 (resp. x_{3t}) dans T_1 (resp. T_2) des arêtes entre S_1 et T_1 (resp. S_2 et T_2).

On montre que $|V_i| \geqslant k-1$ d'où

$$n(G) \geqslant 3t + 3 + 2(\delta - 1) + (t-1)(\delta - 2) + 2(k-1),$$

$$n(G) \geqslant (t+1)(\delta + 1) + 2k.$$

Démonstration du théorème (continuée).

$$g_1(D, \delta, k) - g_2(D, \delta, k) = (t-1)(3k - \delta - 1).$$

(a) Si $k \leqslant \delta \leqslant 3k - 1$, $g_1(D, \delta, k) \geqslant g_2(D, \delta, k)$.

Le graphe G_1 (Fig. 2) appartient à $\mathscr{G}(D, \delta, k)$ et il a $g_1(D, \delta, k)$ sommets d'où si