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Eckhard Platen · David Heath

A Benchmark Approach to Quantitative Finance

With 199 Figures

Springer

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Preface

In recent years products based on financial derivatives have become an indispensable tool for risk managers and investors. Insurance products have become part of almost every personal and business portfolio. The management of mutual and pension funds has gained in importance for most individuals. Banks, insurance companies and other corporations are increasingly using financial and insurance instruments for the active management of risk. An increasing range of securities allows risks to be hedged in a way that can be closely tailored to the specific needs of particular investors and companies. The ability to handle efficiently and exploit successfully the opportunities arising from modern quantitative methods is now a key factor that differentiates market participants in both the finance and insurance fields. For these reasons it is important that financial institutions, insurance companies and corporations develop expertise in the area of *quantitative finance*, where many of the associated quantitative methods and technologies emerge.

This book aims to provide an *introduction to quantitative finance*. More precisely, it presents an introduction to the mathematical framework typically used in financial modeling, derivative pricing, portfolio selection and risk management. It offers a unified approach to risk and performance management by using the *benchmark approach*, which is different to the prevailing paradigm and will be described in a systematic and rigorous manner.

This approach uses the *growth optimal portfolio* as numeraire and the real world probability measure as pricing measure. The existence of an equivalent risk neutral probability measure is not required, which is one of the aspects distinguishing the approach in this book from other more conventional texts in the area. It is our experience that many practitioners find the use of the *real world* probability measure attractive for *pricing* because it is natural and pricing can still be carried out even under circumstances when a risk neutral probability measure cannot exist.

We have attempted to write a multi-purpose book that provides information and methods for a wide range of professionals, researchers and graduate students. It is designed for three groups of readers. In the first instance it should provide useful information to financial analysts and practitioners in the investment, banking and insurance industries. Other professionals at financial software companies, hedge funds, consultants, regulatory authorities and government agencies may significantly benefit from using this book. Secondly, the book aims to introduce those with a reasonable basic mathematical background to the area of quantitative finance. Engineers, computer scientists, numerical analysts, physicists, theoretical chemists, biologists, astrophysicists, statisticians, econometricians, actuaries and other readers should be able to gain access to the field through the book. Thirdly, researchers in financial mathematics will find the later parts of the book interesting and possibly challenging. In particular, the monograph aims to stimulate further developments of the benchmark approach.

The material presented is a self-contained introduction that could be part of a coursework masters or PhD program in quantitative finance. The areas of probability and statistics, stochastic calculus, optimization and numerical methods relevant to finance are all introduced. The book has been designed in a modular way with cross references so that it can also be used as a handbook allowing relevant definitions, formulas and results to be easily looked up.

The monograph is divided into fifteen chapters. The first two chapters summarize fundamental results from probability and statistics which are essential for quantitative finance. Some statistical analysis on the log-return distribution of indices is included at the end of Chap. 2.

The Chaps. 3 and 4 introduce stochastic processes. The stochastic calculus needed for financial modeling using stochastic differential equations is presented in Chaps. 5 to 7. Stochastic differential equations with jumps are introduced from a finance perspective. Some of the material goes beyond what can be found in standard textbooks.

In Chap. 8 basic financial derivatives are introduced from a hedging perspective. European call and put options are priced via the corresponding Black-Scholes partial differential equation. The sensitivities of these option prices to movements in parameter values are studied. Hedge simulations are performed, which illustrate derivative pricing and hedging.

Chapter 9 presents various alternative pricing methodologies. First, the concept of *real world pricing* is introduced. Several other pricing methods are shown to be special cases of real world pricing. These include actuarial pricing, risk neutral pricing and pricing under change of numeraire. The existence of an equivalent risk neutral probability measure is *not* required under the benchmark approach. The chapter concludes by introducing the Girsanov theorem, the Bayes rule and the Feynman-Kac formula.

Chapter 10 develops a unified modeling framework for continuous financial markets under the benchmark approach. It presents a range of new concepts and ideas that do not fit under the presently prevailing approaches. A *diversification theorem* is derived, which shows under some regularity condition that diversified portfolios approximate the growth optimal portfolio. This allows us to interpret a diversified market index as a proxy for the growth optimal portfolio.

Chapter 11 derives results on portfolio optimization via the maximization of Sharpe ratios. The capital asset pricing model (CAPM), the Markowitz efficient frontier, two fund separation and results on expected utility maximization, utility indifference pricing, derivative pricing and hedging are also presented in this chapter.

The modeling of stochastic volatility of stock market indices under the benchmark approach is discussed in Chap. 12. This analysis includes the pricing of index derivatives under models that do not admit an equivalent risk neutral probability measure. More general volatility models than those permitted under the standard risk neutral approach are covered.

In Chap. 13 it is shown that the discounted growth optimal portfolio follows the dynamics of a time transformed squared Bessel process of dimension four. Making the drift of the discounted growth optimal portfolio a function of time, yields the *minimal market model*. Derivative prices which follow under this parsimonious model appear to be rather realistic. Long term derivatives can be realistically priced. These prices deviate significantly from those obtained under risk neutral pricing because the hypothetical risk neutral measure has after several years a total mass that is significantly less than one. Extensions of the minimal market model with random scaling are considered.

In Chap. 14 models are analyzed that permit jumps to model event risk. Most of the results of previous chapters are generalized to jump diffusion markets. Two market models illustrate differences in derivative pricing under the standard risk neutral and the benchmark approach.

Finally, in Chap. 15 a brief introduction is given from a unifying perspective to basic numerical methods for quantitative finance. This introduction covers scenario simulation, Monte Carlo simulation, tree based methods and finite difference methods. A binomial tree method is developed for the benchmark approach and finite difference methods are explained as numerical methods for systems of coupled ordinary differential equations.

Selected *exercises* at the end of each chapter should enable the reader to further develop skills and test the understanding of the subject. *Solutions* to these exercises are included at the end of the book. The material can be taught at different levels. The first sections in most chapters provide a less technical presentation of the subject. At the end of some sections or chapters (*)-subsections or (*)-sections have been included. These are more technical in nature and are usually not necessary for a first reading.

The formulas are numbered according to the chapter and section where they appear. Assumptions, theorems, lemmas, definitions and corollaries are numbered sequentially in each section. The most common notations are listed at the beginning of the book and an *index of keywords* is given at its end. Some readers may find the *author index* at the end of the book useful.

Substantial work is involved in studying the material presented. This should not be underestimated by the reader. Actively solving exercises is strongly recommended. The reward for this demanding work will be a sound understanding of essential methods in quantitative finance with an emphasis on the benchmark approach.

The authors would like to thank several colleagues and PhD students for many valuable suggestions on the manuscript, including Nicola Bruti-Liberati, Carl Chiarella, Boris Choy, Morten Christensen, Marc Craddock, Ernst Eberlein, Robert Elliott, Kevin Fergusson, Chris Heyde, John van der Hoek, Hardy Hulley, Monique Jeanblanc, Leah Kelly, Truc Le, Shane Miller, Alex Novikov, Alun Pope, Wolfgang Runggaldier and Marc Yor. The authors would like to express their deep gratitude to Katrin Platen, who organized all technical work on the book, in particular, many figures. She carefully and patiently type set the countless versions of the extensive manuscript. Finally, we like to thank Catriona Byrne from Springer Verlag for her excellent work and for encouraging us to write this book.

It is greatly appreciated if readers could forward any errors, misprints or suggested improvements to: eckhard.platen@uts.edu.au

The interested reader is likely to find updated information about the benchmark approach, as well as, teaching material related to the book on the webpage of the first author under "Benchmark_Approach":

`http://www.business.uts.edu.au/`
`finance/staff/Eckhard/Benchmark_Approach.html`

Sydney,
March 2006

Eckhard Platen
David Heath

Contents

Basic Notation

μ_X	mean of X; 21, 22
σ_X^2, $\mathrm{Var}(X)$	variance of X; 23, 24
β_X	skewness of X; 25
κ_X	kurtosis of X; 26
$\underline{\kappa}_X$	excess kurtosis; 28
$\mathrm{Cov}(X, Y)$	covariance of X and Y; 39
$\inf\{\cdot\}$	greatest lower bound; 94, 129
$\sup\{\cdot\}$	smallest upper bound; 61, 79, 94, 128, 129
$\max(a, b) = a \vee b$	maximum of a and b; 170
$\min(a, b) = a \wedge b$	minimum of a and b; 170
$\boldsymbol{x}^\top$	transpose of a vector or matrix $\boldsymbol{x}$; 40
$\boldsymbol{x} = (x^1, x^2, \ldots, x^d)^\top$	column vector $\boldsymbol{x} \in \Re^d$ with ith component x^i; 44
$\lvert\boldsymbol{x}\rvert$	absolute value of $\boldsymbol{x}$ or Euclidean norm; 20, 22, 49
$\boldsymbol{A} = [a^{i,j}]_{i,j=1}^{k,d}$	$(k \times d)$-matrix $\boldsymbol{A}$ with ijth component $a^{i,j}$; 40
$\det(\boldsymbol{A})$	determinant of a matrix $\boldsymbol{A}$; 40
$\boldsymbol{A}^{-1}$	inverse of a matrix $\boldsymbol{A}$; 41, 46
$(\boldsymbol{x}, \boldsymbol{y})$	inner product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$; 49
$\mathcal{N} = \{1, 2, \ldots\}$	set of natural numbers; 5
∞	infinity; 2

(a, b)	open interval $a < x < b$ in $\Re$; 8
$[a, b]$	closed interval $a \leq x \leq b$ in $\Re$; 12
$\Re = (-\infty, \infty)$	set of real numbers; 8
$\Re^+ = [0, \infty)$	set of nonnegative real numbers; 39
$\Re^d$	d-dimensional Euclidean space; 38
Ω	sample space; 4
$\emptyset$	empty set; 4
$A \cup B$	the union of sets A and B; 4
$A \cap B$	the intersection of sets A and B; 4
$A \backslash B$	the set A without the elements of B; 124, 258, 359
$\mathcal{E} = \Re \backslash \{0\}$	$\Re$ without origin; 124, 564
$[X, Y]_t$	covariation of processes X and Y at time t; 178
$[X]_t$	quadratic variation of process X at time t; 172
$n! = 1 \cdot 2 \cdot \ldots \cdot n$	factorial of n; 10, 62
$[a]$	largest integer not exceeding $a \in \Re$; 522
i.i.d.	independent identically distributed; 55
a.s.	almost surely; 6, 56
f'	first derivative of $f : \Re \rightarrow \Re$; 14
f''	second derivative of $f : \Re \rightarrow \Re$; 219, 421
$f : Q_1 \rightarrow Q_2$	function f from Q_1 into Q_2; 8
$\frac{\partial u}{\partial x^i}$	ith partial derivative of $u : \Re^d \rightarrow \Re$; 39
$\left(\frac{\partial}{\partial x^i}\right)^k u$	kth order partial derivative of u with respect to x^i; 39
$\exists$	there exists; 128
$F_X(\cdot)$	distribution function of X; 8
$f_X(\cdot)$	density function of X; 11
$\phi_X(\cdot)$	characteristic function of X; 35
$\mathbf{1}_A$	indicator function for event A to be true; 9

$N(\cdot)$	Gaussian distribution function; 14
$\Gamma(\cdot)$	gamma function; 15
$\Gamma(\cdot;\cdot)$	incomplete gamma function; 15
$\pmod c$	modulo c; 548
$\mathcal{A}$	collection of events, sigma-algebra; 5
$\underline{\mathcal{A}}$	filtration; 162
$E(X)$	expectation of X; 21, 22
$E(X \mid \mathcal{A})$	conditional expectation of X under $\mathcal{A}$; 32, 33
$P(A)$	probability of A; 4
$P(A \mid B)$	probability of A conditioned on B; 6
$\in$	element of; 1
$\notin$	not element of; 4
$\neq$	not equal; 5
$\approx$	approximately equal; 72, 169
$a \ll b$	a is significantly smaller than b; 426, 515
$\lim_{N\to\infty}$	limit as N tends to infinity; 2
$\liminf_{N\to\infty}$	lower limit as N tends to infinity; 93, 94
$\limsup_{N\to\infty}$	upper limit as N tends to infinity; 93, 94
$\imath$	square root of -1, imaginary unit; 35, 149
$\delta(\cdot)$	Dirac delta function at zero; 143, 146
$\boldsymbol{I}$	unit matrix; 44
$\mathrm{sgn}(x)$	sign of $x \in \Re$; 42
$\mathcal{L}_T^2$	space of square integrable, progressively measurable functions on $[0,T] \times \Omega$; 191
$\mathcal{B}(U)$	smallest sigma-algebra on U; 124
$\ln(a)$	natural logarithm of a; 1
MM	Merton model; 252
MMM	minimal market model; 251

EWI	equi-value weighted index; 397
MSCI	Morgan Stanley capital weighted world stock accumulation index; 332
ODE	ordinary differential equation; 151, 239
SDE	stochastic differential equation; 207, 235, 237
PDE	partial differential equation; 143
PIDE	partial integro differential equation; 358
WSI	world stock index; 399
$I_\nu(\cdot)$	modified Bessel function of the first kind with index ν; 16
$K_\lambda(\cdot)$	modified Bessel function of the third kind with index λ; 17, 18
$\mathcal{V}$	set of nonnegative portfolios; 373
$\mathcal{V}^+$	set of strictly positive portfolios; 369
$\bar{\mathcal{V}}^+_{S_0}$	set of strictly positive, discounted fair portfolios with initial value S_0; 419

1

Preliminaries from Probability Theory

This chapter reviews some important results from probability theory and fixes notation. First we introduce discrete and continuous random variables and their distributions. Then we discuss functionals of random variables such as moments. Furthermore, we introduce certain classes of distributions and also multivariate distributions together with copulas.

1.1 Discrete Random Variables and Distributions

In financial markets one can observe the prices of assets such as stocks, commodities, currencies, futures, bonds etc. It is a challenge to model these random quantities in a satisfactory manner.

Log-Returns

Let us assume that we observe an asset price at times $t_i = i\Delta$ for $i \in \{0, 1, \ldots\}$ with time step size $\Delta > 0$. The time Δ between two successive observations is typically the length of one day. If X_{t_i} denotes the asset price at time t_i, then the *log-return* R_{t_i} at this time is defined as

$$R_{t_i} = \ln(X_{t_{i+1}}) - \ln(X_{t_i}) = \ln\left(\frac{X_{t_{i+1}}}{X_{t_i}}\right) \tag{1.1.1}$$

for $i \in \{0, 1, \ldots\}$.

We define the daily log-return of an asset price as the daily increment of the natural logarithm of this price because, as we shall see later on, this reflects well the growth nature of economies and financial markets. Typically log-returns exhibit considerable variability.

We focus in this book on the modeling of log-returns while we introduce the basic concepts of probability, statistics, stochastic processes, stochastic calculus and stochastic differential equations. It will turn out that stochastic differential equations provide an ideal mathematical framework for the modeling of financial quantities. In this context log-returns will also allow us to apply the powerful tools of stochastic calculus. This is not so conveniently achieved when using, so-called, *returns* that are of the form

$$\tilde{R}_{t_i} = \frac{X_{t_{i+1}} - X_{t_i}}{X_{t_i}}$$

and closely approximate log-returns when these are small. As we shall see, log-returns are more tractable in continuous time.

Relative Frequencies and Probabilities

Let us interpret an asset's log-return R_{t_i} as the *outcome* of an experiment based on observations of the data. Suppose, for simplicity, that we classify the log-returns as strictly negative, zero or positive. We denote these *elementary outcomes* or *states* by ω_1, ω_2, ω_3, indicating that we observe a negative, zero or strictly positive log-return, respectively. We call the set of outcomes or states $\Omega = \{\omega_1, \omega_2, \omega_3\}$ the *sample space* for our experiment.

If we repeat our experiment N times, that is, we observe for a stock daily log-returns on N different days, and count the number $N(\omega_i)$ of times, that the outcome ω_i occurs, we can form the *relative frequency*

$$f_i(N) = \frac{N(\omega_i)}{N}.$$

For smaller N this number usually varies considerably. As N becomes larger, our experience would indicate that the relative frequency should approach a limit p_i, written as

$$\lim_{N\to\infty} f_i(N) = p_i,$$

which we call the *probability* of outcome ω_i.

To illustrate the above example let us look at the daily IBM share price in US dollars over the period from 1977 until 1997, which is shown in Fig. 1.1.1. The corresponding log-returns are plotted in Fig. 1.1.2. In Fig. 1.1.3 we then display the relative frequencies $f_1(t_i)$, $f_2(t_i)$ and $f_3(t_i)$, $i \in \{0, 1, \ldots\}$, of negative, zero and strictly positive log-returns, respectively, during the time period. Note that after some wild fluctuations for small time t, at the beginning of the period, the relative frequency for negative log-returns stabilizes around a value close to $p_1 = 0.465$. Similarly, we obtain at the end of the period a value $p_3 = 0.463$ for the relative frequency of strictly positive log-returns. The value $p_2 = 0.072$ is then obtained for the rather small probability of zero log-returns. Clearly, we have $0 \leq p_i \leq 1$ for each $i \in \{1, 2, 3\}$ and $\sum_{i=1}^{3} p_i = 1$, that is, the probabilities p_1, p_2 and p_3 add up to one.

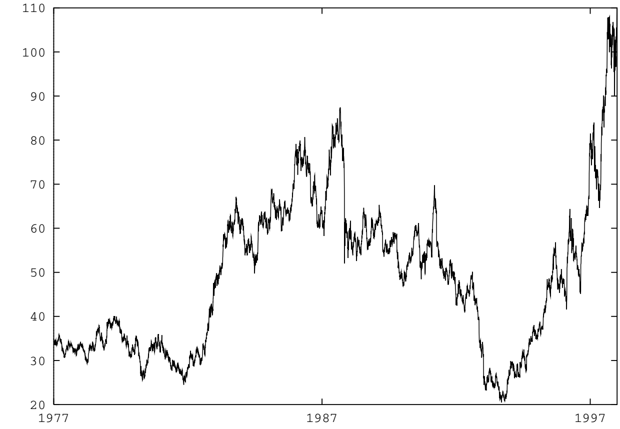

Fig. 1.1.1. IBM share price from 1977 until 1997

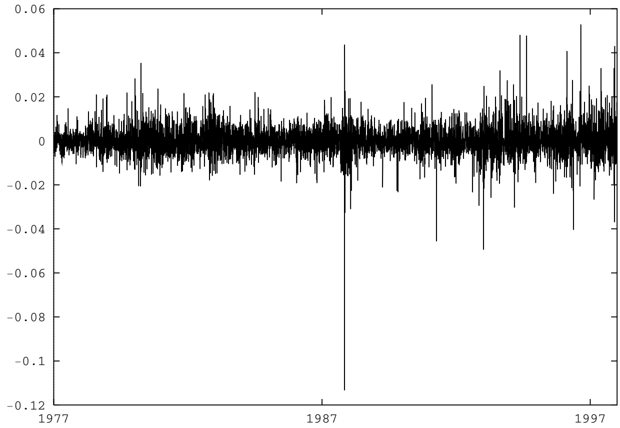

Fig. 1.1.2. Log-returns of IBM stock

Probability Space

To analyze a model one is often interested in combinations of outcomes. We call such a combination an *event* if we can identify it either by its occurrence or its non-occurrence. Obviously, if a subset A of the set of outcomes Ω is an event, then its *complement* $A^c = \{\omega_i \in \Omega : \omega_i \notin A\}$, which denotes the set of all ω_i from the sample space Ω that do not belong to the set A, must also be an event. In the case of the above example we might consider the event $A = \{\omega_1, \omega_2\}$ that corresponds to the occurrence of either a negative or zero log-return. The complement of this event is then $A^c = \{\omega_i \in \Omega : \omega_i \notin \{\omega_1, \omega_2\}\} = \{\omega_3\}$. This is the event $\{\omega_3\}$ of a strictly positive log-return.

In particular, the whole sample space Ω is an event, which is called the *sure event* since one of its outcomes must always occur. The complement of Ω

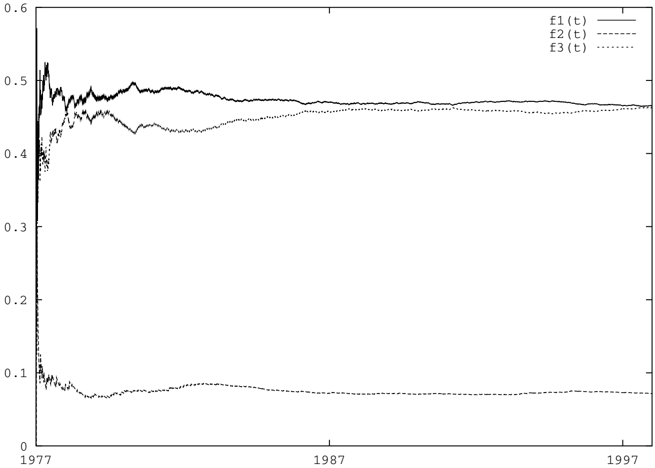

Fig. 1.1.3. Relative frequency over time

is the *empty set* $\emptyset$, which is also defined as an event but never occurs. If A and B are events, then the event $A \cup B$ occurs if either A or B occurs, whereas the event $A \cap B$ occurs if both A and B occur. With $A = \{\omega_1, \omega_2\}$, as in our example, and the event $B = \{\omega_2\}$ indicating a zero log-return we note that $A \cup B = \{\omega_1, \omega_2\} \cup \{\omega_2\} = \{\omega_1, \omega_2\}$ stands for an event consisting of either negative or zero log-returns and $A \cap B = \{\omega_1, \omega_2\} \cap \{\omega_2\} = \{\omega_2\}$ is the event which indicates only a zero log-return.

In the above discussion we have only mentioned experiments with a finite number of outcomes. However, the introduction of probabilities based on an infinite set of outcomes and the use of relative frequencies to define probabilities can lead to conceptual subtleties and other mathematical problems. To resolve these difficulties, Kolmogorov developed in the late 1920s an axiomatic approach to probability theory. In this approach the probabilities represent numbers assigned to corresponding events. In what follows we shall employ this axiomatic framework.

Let us denote by $P(A)$ the *probability* of the occurrence of an event A that is taken from the *collection of events* $\mathcal{A}$ that corresponds to the sample space Ω. Then from corresponding properties of relative frequencies we would expect these probabilities to satisfy the following relationships

$$0 \leq P(A) \leq 1, \tag{1.1.2}$$

$$P(A^c) = 1 - P(A), \tag{1.1.3}$$

$$P(\emptyset) = 0, \qquad P(\Omega) = 1, \tag{1.1.4}$$

and

$$P(A \cup B) = P(A) + P(B) \tag{1.1.5}$$

if A and B are exclusive, that is $A \cap B = \emptyset$ for events A and B taken from $\mathcal{A}$.