DIFFERENTIAL GEOMETRY AND TOPOLOGY OF CURVES

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Preface

Differential geometry is a wide domain of modern mathematics, whose significance is increasing at present. One of its origins is in the theory of curves. Everybody who wishes to study geometric problems has to begin by studying the theory of curves, where exact definitions, notions, and invariant characteristics are introduced for the first time. Here the initial geometric intuition is formed and then it is developed in the studying of surfaces theory and the geometry of submanifolds.

There exist good and extensive monographs devoted to special curves, but the problems of the general theory are not presented. On the other hand, many interesting and important questions on curves are not discussed, in most cases, in the courses on differential geometry in universities.

This book is devoted to the general topics of the geometry of curves as well as to some particular results. Presentation begins with important definitions, including definition of a curve. We also introduce basic notions by using sufficiently accessible language. Next, we discuss properties 'in the large' of the curves in Euclidean space, which were presented earlier in scientific articles only.

For a plane curve, the conditions on the curvature of a closed curve as a function of the arc length are well known. Therefore Efimov, Fenchel and other geometers state the following problem: what are the necessary and sufficient conditions on the curvature and torsion of a space curve in order for the curve to be closed? Probably effective conditions do not exist. But this question connects with other interesting questions. In this book we investigate problems for special classes of curves, give the working method to obtain the conditions for closed polygonal curves, and give the proof of the Bakelman–Werner theorem on necessary and sufficient conditions of the boundedness for curves with periodic curvature and torsion. We investigate the question of the connection between curvature and torsion for curves which we know are closed – curves of the trigonometrical type.

An important geometrical characteristic of a curve is its indicatrix of tangents which we construct in the following way. Let P be a point on curve Γ and $\tau(P)$ a unit tangent vector of Γ at point P. We translate $\tau(P)$ in such a manner that the origin of $\tau(P)$ coincides with the origin O of Cartesian coordinates. The set of end points of translated vectors $\tau(P)$ is called the spherical indicatrix of tangents of the curve Γ. For a closed regular curve this set is not arbitrary: it cannot lie on a hemisphere. This circumstance was probably first noticed by Poznjak. It was observed that the mentioned necessary condition is also sufficient. In this book we give the proof of the Vygodsky theorem: if γ is a closed curve on the unit sphere such that γ does not lie on any hemisphere, then γ is the spherical indicatrix of some space curve. The short proof given in the book belongs to the well-known mathematician M. Krein.

The wonderful French mathematician and astronomer Ch. Delaunay proposed a problem to obtain the curve of constant curvature $k = 1$ which passes through given two points and has the smallest or the greatest length. We give the solution of this problem by K. Weierstrass. Later Schwarz formulated a theorem that the length of such a curve cannot lie in some interval, but did not publish any proof. Then Schur, motivated by Hilbert, proved this theorem by using the twisting of a plane curve. By the twisting of γ Schur meant a transformation of γ preserving the curvature and the length of γ. So, the twisting here is some process. Schur proved that if a plane curve γ with end points P and T forms, together with the span PT, a closed convex curve, then as a result of the twisting the length of the span PT only increases.

In 1929, Fenchel proved a theorem that the integral of curvature for a closed curve is not less than 2π. Borsuk proposed that for a knotted curve this integral is not less than 4π. Fery and Milnor proved his assumption almost simultaneously in 1949–1950. Later this theorem was generalized for n-dimensional submanifolds by Chern, Lachof, Ferus and others. In 1995, V. Gorkavy suddenly obtained a generalization of the Fenchel inequality in a new direction – for the higher curvatures of a curve in n-dimensional Euclidean space.

In accordance with the title of the book, we pay much attention to topological questions. First, we discuss Gauss's classical integral for two linked curves. We observe the links between two infinitesimally close curves and prove the formulas of Calagareanu and White, which have application in biology. Here also twisting arose, but as a number characterizing the form of a curve. Later we discuss knots and knot groups, and give proof of the Pontryagin–Frankl theorem that every knot is the boundary of some oriented surface. In a geometrical way we construct Alexander's polynomial. It is the knot invariant and the proof is founded on the three kinds of changes in the structure of the knot diagram. A list of the simplest knots and their Alexander polynomials are given. In 1983, Jones constructed a new polynomial using the braid theory and some Markov theorems. This polynomial was a great surprise to topologists. In 1985, almost simultaneously six mathematicians constructed a new polynomial depending on three arguments. It was called the HOMFLY polynomial. It is possible to obtain two previous polynomials from the HOMFLY polynomial. We give the method for calculating the HOMFLY polynomial, moving from simple knots and links to more complicated ones.

A very interesting direction for the application of differential geometry and knot theory arose in the biology of DNA molecules. Because biologists obtained closed DNA molecules, the following question began to concern them: how could the process of replication of DNA molecules take place, if two chains of nucleotides are very closely linked? The chapter devoted to this question and the list of works give readers the possibility of acquainting themselves with important research in this direction.

Three chapters are devoted to the theory of curves in n-dimensional Euclidean space. We give the definition and formulas for the calculation of all curvatures of curves, and obtain the canonical form for curves with all constant curvatures. Its behavior is essentially different in spaces of odd and even dimensions.

The curves are applied in mathematics and technology, so investigation of them is very relevant at present.

I would like to express my thanks to V. Gorkavy for the translation and other assistance. I am very grateful to Dr R. Rennie for his useful indications on the works of Jones and Witten. I am also grateful to Professor Nigel Hitchin who encouraged me to introduce the section on DNA.

1 Definition of a Curve

The notion of a curve is one of the most important notions in differential geometry. In antiquity this notion had no explicit mathematical definition. Euclid, for example, defines a curve as a "length without width". At this time many wonderful and interesting curves were discovered and studied; however, the idea of a general curve remained at a trivial, obvious level. Further technological progress required the development of natural science, especially the evolution of mechanics and mathematics. It was necessary to understand clearly the foundations of mathematics and, in particular, to construct an accurate definition of a curve. The coordinates method proposed by Descartes prepared the way for a general definition of curves; mathematicians contemporary to Descartes defined a plane curve given by an equation $\Phi(x, y) = 0$ as a set of points such that their Cartesian coordinates satisfy this equation. Another idea arose in mechanics: a curve is imagined as the trace of some moving point, whose coordinates depend on the time t. Jordan proposed the following definition: a space curve is a set of points whose Cartesian coordinates x, y, z are continuous functions

$$\begin{aligned} x &= \varphi_1(t), \\ y &= \varphi_2(t), \\ z &= \varphi_3(t), \end{aligned} \tag{1.1}$$

of some parameter t varying inside a real axis segment (a, b); in other words, a curve is defined as the image of a real axis segment under a continuous map into the space. This definition seemed to be natural, but in 1890 Peano constructed a continuous map of a segment (a, b) into the space such that the image of (a, b) under this map covered the whole square (we will consider Peano's example in one of the following chapters). In 1897 Klein remarked: "What is an arbitrary curve?... One may say that at present in mathematics there exists no more dark and more indefinite notion than the mentioned one. The object, which we call a mass curve, is a strip, whose length is sufficiently great with respect to another strip's measures. But for a curve to be a subject of strong mathematical consideration, we must idealize a curve in the same way as a point is idealized. And here some difficulties arise... Let us turn to a proposition playing an essential role in Riemann's investigations into foundations of geometry: the space can be viewed as a three-dimensional continuous manifold... We start from a construction of some scale on a mass straight line; then we decompose the scaled line into smaller parts and continue this operation until it is realizable. After that we make the most important step from an experience to an axiom: *we postulate that the correspondence between points and real numbers is valid not only empirically, but also absolutely....*"

We remark that Veronese considered a geometry with the following assumption: on the real axis there exist numbers different from the rational and irrational numbers; but it seems that this supposition does not lead to essential geometrical statements and there is no natural foundation for it at present. Another extreme way of looking at space is proposed by discrete geometry.

Riemann noted: "The question of the validity of the assumptions of geometry 'in the small' is closely connected with the question of inner sources of metric relations in the space. Certainly, this question belongs to the theory of space and we must take into account that in the case of a discrete manifold the principle of metric relations is contained in the notion of this manifold, whereas in the case of a continuous manifold we have to seek it in some other place. From this it follows that either the real space is a discrete manifold, or we must explain the appearance of metric relations by something exterior...."

In modern differential geometry a slightly modified definition of Jordan is used. First, we will give *the definition of an elementary curve*. Let φ be a map of a segment (a, b) of the real axis into the Euclidean space; we denote by γ the image of (a, b) under φ. The map φ is called continuous at a point $X \in (a, b)$, if for any positive ϵ there exists a positive δ such that the following condition is fulfilled: if a point $Y \in (a, b)$ satisfies the inequality $|X - Y| \leq \delta$, then the distance between the points $\varphi(X)$, $\varphi(Y)$ is less than ϵ. The map φ is said to be continuous if it is continuous at each point of the segment (a, b). The map φ is called one-to-one if the pre-image of any point $P \in \gamma$ consists of one point. If γ is one-to-one, then one can construct the map φ^{-1} inverse to φ. The domain of definition of φ^{-1} is γ; the map φ^{-1} assigns to a point $P \in \gamma$ its pre-image under the map φ; in other words, if $P \in \gamma$ and $P = \varphi(X)$, then $\varphi^{-1}(P) = X$ by definition. The inverse map φ^{-1} is continuous at $P \in \gamma$ if for any positive ϵ there exists a positive δ such that the following condition is fulfilled: if $Q \in \gamma$ and the distance between P, Q is less than δ, then $|\varphi^{-1}(P) - \varphi^{-1}(Q)| \leq \epsilon$.

Definition *A set of the Euclidean space is called an elementary curve if this set is the image of an interval of the real axis under a one-to-one continuous map, whose inverse map is continuous too.*

We remark that a one-to-one continuous map, whose inverse map is continuous, is called *a homeomorphism* or *a topological map*.

Suppose the considered set γ is an elementary curve. A point $X \in (a,b)$ can be viewed as a real number $t \in (a,b)$; under the map φ some point $P = \varphi(X)$ is assigned to the point X. Assume that Cartesian coordinates x, y, z are fixed in the space. Then one can consider the coordinates of the point P as functions of the real parameter t:

$$\begin{aligned} x &= \varphi_1(t), \\ y &= \varphi_2(t), \\ z &= \varphi_3(t). \end{aligned} \tag{1.1}$$

These equalities are called a *parametric representation* of the curve γ; sometimes γ is said to be *parametrized by the parameter* $t \in (a,b)$. If the interval (a,b) is topologically mapped onto an interval (c,d), we can view t as a monotone continuous function of a parameter $\tau \in (c,d)$. Since the composition of the topological maps $(c,d) \to (a,b)$ and $(a,b) \to \gamma$ is a homeomorphism, the curve γ can be parametrized by the parmeter τ. Thus γ can be presented as

$$\begin{aligned} x &= \varphi_1(t(\tau)) = f_1(\tau), \\ y &= \varphi_2(t(\tau)) = f_2(\tau), \\ z &= \varphi_3(t(\tau)) = f_3(\tau) \end{aligned}$$

as well as in form (1.1). We note that an elementary curve can be very complicated. For example, the projection of some elementary curve into a plane can be Peano's curve covering a square.

Definition *An elementary curve γ is called a simple curve if γ is the image either of a segment of the real axis, or of a circle under a homeomorphism. The image of a circle under a homeomorphism is called a closed Jordan curve.*

Connection property of curves

We will prove the connection property of curves, which is one of the most important properties of curves. First, we say that a point X_0 of the Euclidean space is a *limit point* for a set M if in any neighborhood of X_0 there exists a point of M. The set M is said to be connected if it cannot be decomposed into two disjoint subset M_1, M_2 such that each of M_1, M_2 does not contain limit points of the other. (We note that these definitions are valid not only for the sets of Euclidean space, but also for the subsets of any set of points, where the notion of neighborhoods is defined correctly.)

Let us prove that a segment (a,b) of the real axis is a connected set. Assume the converse; then the segment (a,b) is decomposed into two sets M_1, M_2 such that limit points of M_i do not belong to M_j at $i \neq j$. We observe that M_1 is closed. Indeed, from the assumption it follows that if a point X belongs to M_1, then there exists a neighborhood U_X of X consisting of points of the set M_1. A similar discussion demonstrates that the set M_2 is closed.

Suppose the point b is contained in the set M_2. We denote by c the least upper bound of the points of M_1. The point c is a limit point of M_1, hence it belongs to the closed set M_1 and does not coincide with b. But from the definition of the least upper bound it follows that the points satisfying the inequality $X \geq c$ belong to M_2. Hence c is a limit point of M_2; from this fact it follows that $c \in M_2$. This contradiction proves the connectedness of the segment (a, b).

Now we will demonstrate that the connection property is preserved under any homeomorphism. Let f be a homeomorphism of a connected set M. Suppose the image of M under f is not connected. Then there exist two disjoint subsets A, B of $f(M)$ such that $f(M) = A \cup B$ and each of A, B does not contain limit points of the other. We consider the sets $f^{-1}(A), f^{-1}(B)$ situated in M. Since M is connected, there exists a point $X_0 \in f^{-1}(A)$ which is a limit point of $f^{-1}(B)$. Because of the continuity of f, for an arbitrary neighborhood V_{Y_0} of the point $Y_0 = f(X_0) \in A$ there exists a neighborhood U_{X_0} of X_0 such that $f(U_{X_0}) \subset V_{Y_0}$. Since X_0 is a limit point of $f^{-1}(B)$, the neighborhood V_{Y_0} contains a point of B. Hence Y_0 is a limit point of B that contradicts the assumption. Thus the image of M under f is connected as well as M. Now it is easy to see that *a simple curve is connected.*

In the next chapters we will consider curves with self-intersection points. So, for the sequel we need to construct a more general notion of a curve. Let an interval (a, b) (or a circle) be continuously mapped into the space in such a way that for any point of (a, b) there exists a neighborhood whose image is an elementary curve; then the image γ of the whole interval (a, b) is called *a general curve*. The general curve γ can contain a point corresponding to different points of (a, b); such a point is said to be *a self-intersection point* of γ. When a point of the interval moves from a to b, the corresponding moving point on the curve γ passes through any self-intersection point at least twice.

2 Vector-valued Functions Depending on Numerical Arguments

Let a real parameter t vary in an interval (a, b). If to each value $t \in (a, b)$ we assign a vector $r(t)$, then we say that *a vector-valued function $r(t)$ with argument $t \in (a, b)$* is given. Assume Cartesian coordinates x, y, z are fixed; then the representation of the vector-valued function $r(t)$ is equivalent to the representation of three scalar (real-valued) functions $x(t)$, $y(t)$, $z(t)$. We can write $r(t) = \{x(t), y(t), z(t)\}$, but the brief notation $r(t)$ is more convenient. One can define many notions connected with vector-valued functions similarly to notions corresponding to the usual scalar functions.

First, we can define the *limit of $r(t)$ as $t \to t_0$*. A vector r_0 is called the *limit of the vector-valued function $r(t)$ as $t \to t_0$* if the length of the vector $r(t) - r(t_0)$ tends to zero as $t \to t_0$. Here we write

$$\lim_{t \to t_0} r(t) = r_0.$$

It is clear that the vector-valued function $r(t)$ has a limit iff each one of the functions $x(t)$, $y(t)$, $z(t)$ has a limit as $t \to t_0$. Limits of vector-valued functions have the same properties as limits of scalar functions.

The vector-valued function $r(t)$ is said to be *continuous* at t_0 iff $\lim_{t \to t_0} r(t)$ is equal to the value of $r(t)$ at t_0:

$$\lim_{t \to t_0} r(t) = r(t_0).$$

Differentiating the components $x(t)$, $y(t)$, $z(t)$ of $r(t)$, we obtain a function called the *derivative* of $r(t)$ and denoted by

$$r'(t) = \{x'(t), y'(t), z'(t)\}.$$

Moreover, one can define the derivative $r'(t)$ in the same way as derivatives of scalar functions. Precisely, the derivative $r'(t)$ of the vector-valued function $r(t)$ we call the limit:

$$\lim_{t \to t_0} \frac{r(t) - r(t_0)}{t - t_0}.$$

If this limit exists, then $r(t)$ is called *differentiable*. The differentiation of vector-valued functions has the same properties as the differentiation of scalar functions. The sum of the derivatives of two functions is equal to the derivatives of the sum of these two functions:

$$(\rho_1(t) + \rho_2(t))' = \rho_1'(t) + \rho_2'(t).$$

If $f(t)$ is a scalar function and $\rho(t)$ is a vector-valued function, then

$$(f\rho)' = f'\rho + f\rho',$$

i.e. the rule of differentiation in this case is the same as the differentiation rule for products of scalar functions. The differentiation of inner products, vector products, and mixed products of vector-valued functions is computed by the consecutive differentiation of the cofactors. To be precise, if $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ are vector-valued functions, then

$$(\rho_1, \rho_2)' = (\rho_1', \rho_2) + (\rho_1, \rho_2');$$

$$[\rho_1, \rho_2]' = [\rho_1', \rho_2] + [\rho_1, \rho_2'];$$

$$(\rho_1, \rho_2, \rho_3)' = (\rho_1', \rho_2, \rho_3) + (\rho_1, \rho_2', \rho_3) + (\rho_1, \rho_2, \rho_3').$$

For example, let us prove the differentiation rule for the vector product:

$$[\rho_1, \rho_2]'(t_0) = \lim_{t \to t_0} \frac{[\rho_1(t), \rho_2(t)] - [\rho_1(t_0), \rho_2(t_0)]}{t - t_0}$$

$$= \lim_{t \to t_0} \frac{[\rho_1(t) - \rho_1(t_0), \rho_2(t)]}{t - t_0} + \lim_{t \to t_0} \frac{[\rho_1(t_0), \rho_2(t) - \rho_2(t_0)]}{t - t_0}$$

$$= \left[\lim_{t \to t_0} \frac{\rho_1(t) - \rho_1(t_0)}{t - t_0}, \lim_{t \to t_0} \rho_2(t)\right] +$$

$$\left[\lim_{t \to t_0} \rho_1(t), \lim_{t \to t_0} \frac{\rho_2(t) - \rho_2(t_0)}{t - t_0}\right] = [\rho_1'(t_0), \rho_2(t_0)] + [\rho_1(t_0), \rho_2'(t_0)].$$

If $r'(t) = 0$, then $r(t)$ is a constant vector: $r(t) = c$.

By definition, the second derivative $r''(t)$ of $r(t)$ is the derivative of $r'(t)$. By induction, one can define the n-th derivative $r^{(n)}(t)$ of $r(t)$ as the derivative of $r^{(n-1)}(t)$.

Assume that k derivatives of the vector-valued function $r(t)$ exist and are continuous; then we can write Taylor's expansions for the components $x(t)$, $y(t)$, $z(t)$ of $r(t)$:

$$x(t) = x(t_0) + x'(t_0)(t - t_0) + \frac{x''(t_0)}{2!}(t - t_0)^2 + \cdots + o_1(|t - t_0|^k),$$

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \cdots + o_2(|t - t_0|^k),$$

$$z(t) = z(t_0) + z'(t_0)(t - t_0) + \frac{z''(t_0)}{2!}(t - t_0)^2 + \cdots + o_3(|t - t_0|^k).$$

This system of three equations can be rewritten as

$$r(t) = r(t_0) + r'(t_0)(t - t_0) + \frac{r''(t_0)}{2!}(t - t_0)^2 + \cdots + o(|t - t_0|^k),$$

where $o(|t - t_0|^k)$ denotes a vector whose length is an infinitesimal with respect to $|t - t_0|^k$ as $t \to t_0$. We remark that there exists one essential difference between Taylor's expansion of vector-valued functions and Taylor's expansion of scalar functions. If we consider Taylor's expansion for a scalar function $f(t)$, then we have

$$o(|t - t_0|^k) = f^{(k+1)}(\xi)(t - t_0)^{k+1},$$

where ξ is a point situated between t and t_0. For a vector-valued function we cannot write a similar formula for the corresponding infinitesimal vector, because in general for different components of the vector $o(|t - t_0|^k)$ the corresponding points ξ are different. Nevertheless, it is more important that the length of the vector $o(|t - t_0|^k)$ is an infinitesimal with respect to $|t - t_0|^k$.

If we have a continuous vector function $r(t)$, we can define the integral of $r(t)$ as a vector whose components are the integrals of the components $x(t)$, $y(t)$, $z(t)$ of $r(t)$:

$$\int r(t)\,dt = \left\{\int x(t)\,dt, \quad \int y(t)\,dt, \quad \int z(t)\,dt\right\}.$$

The following properties of the defined integral are obvious:

$$\int (r_1(t) + r_2(t))\,dt = \int r_1(t)\,dt + \int r_2(t)\,dt,$$

$$\int \lambda r(t)\,dt = \lambda \int r(t)\,dt, \quad \text{where } \lambda = \text{const}.$$

Moreover, there exist new properties for the integrals of vector-valued functions. If a is a constant vector, then for integrating the inner product $(a, r(t))$ or the vector product $[a, r(t)]$ we can apply the formulas

$$\int (a, r(t))\,dt = \left(a, \int r(t)\,dt\right), \quad \int [a, r(t)]\,dt = \left[a, \int r(t)\,dt\right].$$

For example, let us prove the second formula. The first component of the vector

$$\int [a, r(t)]\, dt$$

is equal to

$$\int (a_2 z(t) - a_3 y(t))\, dt = a_2 \int z(t)\, dt - a_3 \int y(t)\, dt,$$

where a_i are the components of the vector a. We see that on the right-hand side of the last equality we have the first component of the vector

$$\left[a, \int r(t)\, dt\right].$$

Considering the second and third components similarly, we prove the desired formula.

3 The Regular Curve and its Representations

A curve γ is called *C^k-regular*, iff there exists a parametrization of γ such that each component of the position vector $r(t)$ is a C^k-regular function and r'_t does not vanish. The C^1-regular curve is called *smooth* .

The condition $r'_t \neq 0$ is essential for the definition of regular curves. For example, consider the planar curve formed by two rays and represented by the position vector

$$x = at^2, \quad y = bt^2, \quad \text{if} \quad t \leq 0,$$

$$x = ct^2, \quad y = dt^2, \quad \text{if} \quad t \geq 0,$$

where a, b, c, d are constants. If the vectors $\{a, b\}, \{c, d\}$ are linearly independent, the considered line has a singularity at the point $(0, 0)$. At the same time, each component of the position vector is C^1-smooth everywhere. But $r'_t(0) = 0$.

If γ is one-to-one projected onto a segment $[a, b]$ of the x-axis, then there exists one of the simplest representations:

$$r = \{x, y(x), z(x)\}. \tag{3.1}$$

Indeed, let $r(t) = \{x(t), y(t), z(t)\}$ be some representation of γ. Because γ is one-to-one projected onto the segment (a, b) of the x-axis, we can assign to each $x \in (a, b)$ a unique value of the parameter t such that the point $(x, 0, 0)$ is the image of the point $P(t)$ of γ under the projection. Thus t can be viewed as a function of x, i.e. $t = t(x)$. Substituting $t(x)$ into the expressions of the functions $y(x)$, $z(x)$ we obtain:

$$y = y(t(x)) = \tilde{y}(x),$$

$$z = z(t(x)) = \tilde{z}(x).$$

Thus, the position vector of γ parametrized by x has the form (3.1).

When is the curve γ regularly projected onto a segment of the x-axis? Assume that γ is C^1-regular and $x'_t \neq 0$. Consider the function $x = x(t)$. Because $x'_t \neq 0$, the function $x(t)$ is monotone on some interval (a, b) of the x-axis, and there exists the inverse function $t = t(x)$. Since $t'_x = 1/x'_t$, $t(x)$ is C^1-regular. Therefore to each point $x \in (a, b)$ one can assign a unique value of t and a unique point $P(t) \in \gamma$. Thus it is easy to see that γ is projected onto the segment (a, b) of the x-axis. We remark that in this case $\tilde{y}(x)$ and $\tilde{z}(x)$ are C^1-smooth.

Often a planar curve is represented implicitly, i.e. by an equality

$$\Phi(x, y) = 0, \tag{3.2}$$

that is, this curve consists of the points whose coordinates x, y satisfy equality (3.2). But for some functions $\Phi(x, y)$ equation (3.2) has no solution or the solution consists of isolated points. So, it is natural to ask: when does equation (3.2) represent a curve (in the sense of chapter 1)?

Theorem *Let $\Phi(x, y)$ be a C^1-regular function. Assume that a point P with coordinates x_0, y_0 satisfies the equation*

$$\Phi(x, y) = 0$$

and grad $\Phi = \{\Phi_x, \Phi_y\} \neq 0$ *at P. Then the points satisfying (3.2) and situated in some sufficiently small neighborhood of P form a C^1-regular curve.*

Proof We will apply the theorem on an implicit function:

Let a function $\Phi(x, y)$ be defined and C^1-smooth in some neighborhood of a point (x_0, y_0). Assume that $\Phi(x_0, y_0) = 0$ and $\Phi_y(x_0, y_0) \neq 0$. Then there exist $\delta > 0$ and a C^1-regular function $y = y(x)$ defined in the interval $-\delta + x_0 \leq x \leq \delta + x_0$ such that $y(x_0) = y_0$ and $\Phi(x, y(x)) = 0$.

By the assumption, $\operatorname{grad} \Phi(x_0, y_0) \neq 0$. We suppose without loss of generality that $\Phi_y(x_0, y_0) \neq 0$. The assumptions of the theorem on an implicit function are fulfilled. Then the points with coordinates $(x, y(x))$ in some neighborhood of (x_0, y_0) form a curve γ coinciding with the curve represented implicitly by the equation $\Phi(x, y) = 0$. The coordinate x is chosen as a parameter on γ. Because the function $y(x)$ is C^1-regular and $r'_x = \{1, y'_x\} \neq 0$, the curve is smooth. ■

Sometimes it is useful and interesting to consider a family of curves. Here it is convenient to assign to each curve one or more numbers called *parameters of the family*. Then according to the number of parameters the family is called one-parametric, two-parametric and so on. For instance, in the family of straight lines $x = c = const$ on the (x, y)-plane, every straight line is determined by the value of the constant c, i.e. c is the parameter of this family. A second example: the family of circles on the plane is three-parametric.

Let H be a set of planar curves such that for any point P of a domain G on the (u, v)-plane there exists a unique curve $\gamma \in H$ passing through P. We will say that *the set H forms a C^k-regular one-parametric family, iff for any point (u_0, v_0) there exist a neighborhood and a C^k-smooth map $x(u, v), y(u, v)$ of the neighborhood onto the circle*

$x^2 + y^2 < 1$ *with Jacobian* $J \neq 0$ *such that the curves of* H *are transformed into the straight lines* $x = c$.

The level lines $\Phi(u, v) = c$ of a smooth function $\Phi(u, v)$ form a smooth family in a neighborhood of a point (u_0, v_0), if grad $\Phi(u_0, v_0) \neq 0$. Indeed, let for instance $\Phi_u(u_0, v_0) \neq 0$. The map $x = \Phi(u, v)$, $y = v$ has the non-zero Jacobian $J = \Phi_u(u_0, v_0)$ and transforms the level lines $\Phi(u, v) = c$ into the straight lines $x = c$.

Problems

1. Find the projection of the curve

$$x = a\cos t, \quad y = a\sin t, \quad z = bt$$

into the (x, y)-plane. Find the projection into the (x, z)-plane.

2. Does the curve

$$x = t^2 + 1, \quad y = t^2 + 2t + 1, \quad z = t^3 + 1$$

pass through the points $(1, 1, 1)$ and $(1, 0, 0)$?

3. Are the following curves intersecting?

$$\begin{array}{ll} x = t^2 + 1, \quad y = t^2 + 2t + 1, \quad z = t^3 + 1, & x = \tau, \quad y = 2\tau, \quad z = \frac{1}{2}\tau, \\ -\infty \leq t \leq +\infty; & -\infty \leq \tau \leq +\infty. \end{array}$$

4. Find the point where the curve

$$x = t^2 + 2t + 1, \quad y = t^3 + 2, \quad z = t^3 + 1$$

intersects the plane $z = 0$.

5. Show that the curve

$$r(t) = \{a\sin^2 t, \ b\cos t\sin t, \ c\cos t\}$$

is situated on an ellipsoid.

6. Prove that

$$\begin{aligned} x &= x_0 + a\cos t + b\sin t, \\ y &= y_0 - b\cos t + a\sin t, \\ z &= 0 \end{aligned}$$

is the position vector of a circle whose radius is equal to $\sqrt{a^2 + b^2}$.