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Shoshichi Kobayashi

Transformation Groups in Differential Geometry

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Preface

Given a mathematical structure, one of the basic associated mathematical objects is its automorphism group. The object of this book is to give a biased account of automorphism groups of differential geometric structures. All geometric structures are not created equal; some are creations of gods while others are products of lesser human minds. Amongst the former, Riemannian and complex structures stand out for their beauty and wealth. A major portion of this book is therefore devoted to these two structures.

Chapter I describes a general theory of automorphisms of geometric structures with emphasis on the question of when the automorphism group can be given a Lie group structure. Basic theorems in this regard are presented in §§ 3, 4 and 5. The concept of G-structure or that of pseudo-group structure enables us to treat most of the interesting geometric structures in a unified manner. In § 8, we sketch the relationship between the two concepts. Chapter I is so arranged that the reader who is primarily interested in Riemannian, complex, conformal and projective structures can skip §§ 5, 6, 7 and 8. This chapter is partly based on lectures I gave in Tokyo and Berkeley in 1965.

Contents of Chapters II and III should be fairly clear from the section headings. It should be pointed out that the results in §§ 3 and 4 of Chapter II will not be used elsewhere in this book and those of §§ 5 and 6 of Chapter II will be needed only in §§ 10 and 12 of Chapter III. I lectured on Chapter II in Berkeley in 1968; Chapter II is a faithful version of the actual lectures.

Chapter IV is concerned with automorphisms of affine, projective and conformal connections. We treat both the projective and the conformal cases in a unified manner.

Throughout the book, we use *Foundations of Differential Geometry* as our standard reference. Some of the referential results which cannot be found there are assembled in Appendices for the convenience of the reader.

As its title indicates, this book is concerned with the differential geometric aspect rather than the differential topological or homological aspect of the theory of transformation groups. We have confined ourselves to presenting only basic results, avoiding difficult theorems. To compensate for the omission of many interesting but difficult results, we have supplied the reader with an extensive list of references.

We have not touched upon homogeneous spaces, partly because they form an independent discipline of their own. While we are interested in automorphisms of given geometric structures, the differential geometry of homogeneous spaces is primarily concerned with geometric objects which are invariant under given transitive transformation groups. For the convenience of the reader, the Bibliography includes papers on the geometry of homogeneous spaces which are related to the topics discussed here.

In concluding this preface, I would like to express my appreciation to a number of mathematicians: Professors Yano and Lichnerowicz, who interested me in this subject through their lectures, books and papers; Professor Ehresmann, who taught me jets, prolongations and infinite pseudo-groups; K. Nomizu, T. Nagano and T. Ochiai, my friends and collaborators in many papers; Professor Matsushima, whose recent monograph on holomorphic vector fields influenced greatly the presentation of Chapter III; Professor Howard, who kindly made his manuscript on holomorphic vector fields available to me. I would like to thank Professor Remmert and Dr. Peters for inviting me to write this book and for their patience.

I am grateful also to the National Science Foundation for its unfailing support given to me during the preparation of this book.

January, 1972 S. Kobayashi

Contents

I. Automorphisms of G-Structures

1. G-Structures

Let M be a differentiable manifold of dimension n and $L(M)$ the bundle of linear frames over M. Then $L(M)$ is a principal fibre bundle over M with group $\mathrm{GL}(n; \mathbf{R})$. Let G be a Lie subgroup of $\mathrm{GL}(n; \mathbf{R})$. By a *$G$-structure* on M we shall mean a differentiable subbundle P of $L(M)$ with structure group G.

There are very few general theorems on G-structures. But we can ask a number of interesting questions on G-structures, and they are often very difficult even for some specific G. It is therefore essential for the study of G-structures to have familiarity with a number of examples.

In general, when M and G are given, there may or may not exist a G-structure on M. If G is a closed subgroup of $\mathrm{GL}(n; \mathbf{R})$, the existence problem becomes the problem of finding cross sections in a certain bundle. Since $\mathrm{GL}(n; \mathbf{R})$ acts on $L(M)$ on the right, a subgroup G also acts on $L(M)$. If G is a closed subgroup of $\mathrm{GL}(n; \mathbf{R})$, then the quotient space $L(M)/G$ is the bundle with fibre $\mathrm{GL}(n; \mathbf{R})/G$ associated with the principal bundle $L(M)$. It is then classical that the G-structures on M are in a natural one-to-one correspondence with the cross sections

$$M \to L(M)/G$$

(see, for example, Kobayashi-Nomizu [1, vol. 1; pp. 57–58]). The so-called obstruction theory gives necessary algebraic-topological conditions on M for the existence of a G-structure (see, for example, Steenrod [1]).

A G-structure P on M is said to be *integrable* if every point of M has a coordinate neighborhood U with local coordinate system $x^1, \ldots, x^n$ such that the cross section $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ of $L(M)$ over U is a cross section of P over U. We shall call such a local coordinate system $x^1, \ldots, x^n$ *admissible* with respect to the given G-structure P. If $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ are two admissible local coordinate system in open sets U and V respectively, then the Jacobian matrix $(\partial y^i/\partial x^j)_{i,j=1,\ldots,n}$ is in G at each point of $U \cap V$.

Proposition 1.1. *Let* K *be a tensor over the vector space* $\mathbf{R}^n$ *(i.e., an element of the tensor algebra over* $\mathbf{R}^n$*) and* G *the group of linear transformations of* $\mathbf{R}^n$ *leaving* K *invariant. Let* P *be a* G*-structure on* M *and* K *the tensor field on* M *defined by* K *and* P *in a natural manner (see the proof below). Then* P *is integrable if and only if each point of* M *has a coordinate neighborhood with local coordinate system* $x^1, \dots, x^n$ *with respect to which the components of* K *are constant functions on* U.

Proof. We give the definition of K although it is more or less obvious. At each point x of M, we choose a frame u belonging to P. Since u is a linear isomorphism of $\mathbf{R}^n$ onto the tangent space $T_x(M)$, it induces an isomorphism of the tensor algebra over $\mathbf{R}^n$ onto the tensor algebra over $T_x(M)$. Then K_x is the image of K under this isomorphism. The invariance of K by G implies that K_x is defined independent of the choice of u.

Assume that P is integrable and let $x^1, \dots, x^n$ be an admissible local coordinate system. From the construction above, it is clear that the components of K with respect to $x^1, \dots, x^n$ coincide with the components of K with respect to the natural basis in $\mathbf{R}^n$ and, hence, are constant functions.

Conversely, let $x^1, \dots, x^n$ be a local coordinate system with respect to which K has constant components. In general, this coordinate system is not admissible. Consider the frame $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ at the origin of this coordinate system. By a linear change of this coordinate system, we obtain a new coordinate system $y^1, \dots, y^n$ such that the frame $(\partial/\partial y^1, \dots, \partial/\partial y^n)$ at the origin belongs to P. Then K has constant components with respect to $y^1, \dots, y^n$. These constant components coincide with the components of K with respect to the natural basis of $\mathbf{R}^n$ since $(\partial/\partial y^1, \dots, \partial/\partial y^n)$ at the origin belong to P. Let u be a frame at $x \in U$ belonging to P. Since the components of K with respect to u coincide with the components of K with respect to the natural basis of $\mathbf{R}^n$ and, hence, with the components of K with respect to $(\partial/\partial y^1, \dots, \partial/\partial y^n)$, it follows that the frame $(\partial/\partial y^1, \dots, \partial/\partial y^n)$ at x coincides with u modulo G and, hence, belongs to P. q.e.d.

Proposition 1.2. *If a* G*-structure* P *on* M *is integrable, then* P *admits a torsionfree connection.*

Proof. Let U be a coordinate neighborhood with admissible local coordinate system $x^1, \dots, x^n$. Let ω_U be the connection form on $P|U$ defining a flat affine connection on U such that $\partial/\partial x^1, \dots, \partial/\partial x^n$ are parallel. We cover M by a locally finite family of such open sets U. Taking a partition of unity $\{f_U\}$ subordinate to $\{U\}$, we define a desired connection form ω by

$$\omega=\sum_U \pi^* f_U \cdot \omega_U,$$

where $\pi\colon P\to M$ is the projection. q.e.d.

In some cases, the converse of Proposition 1.2 is true. For such examples, see the next section.

Let P and P' be G-structures over M and M'. Let f be a diffeomorphism of M onto M' and $f_*\colon L(M)\to L(M')$ the induced isomorphism on the bundles of linear frames. If f_* maps P into P', we call f an *isomorphism* of the G-structure P onto the G-structure P'. If $M=M'$ and $P=P'$, then an isomorphism f is called an *automorphism* of the G-structure P.

A vector field X on M is called an *infinitesimal automorphism* of a G-structure P if it generates a local 1-parameter group of automorphisms of P.

As in Proposition 1.1, we consider those G-structures defined by a tensor K. Then the following proposition is evident.

Proposition 1.3. *Let K be a tensor over the vector space $\mathbf{R}^n$ and G the group of linear transformations of $\mathbf{R}^n$ leaving K invariant. Let P be a G-structure on M and K the tensor field on M defined by K and P. Then*

(1) *A diffeomorphism $f\colon M\to M$ is an automorphism of P if and only if f leaves K invariant;*

(2) *A vector field X on M is an infinitesimal automorphism of P if and only if $L_X K=0$, where L_X denotes the Lie derivation with respect to X.*

We shall now study the local behavior of an infinitesimal automorphism of an integrable G-structure. Without loss of generality, we may assume that $M=\mathbf{R}^n$ with natural coordinate system $x^1,\dots,x^n$ and $P=\mathbf{R}^n\times G$. Let X be a vector field in (a neighborhood of the origin of) $\mathbf{R}^n$ and expand its components in power series:

$$X=\sum \xi^i \frac{\partial}{\partial x^i}$$

$$\xi^i \sim \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1,\dots,j_k=1}^{n} a^i_{j_1\dots j_k} x^{j_1}\dots x^{j_k},$$

where $a^i_{j_1\dots j_k}$ are symmetric in the subscripts $j_1,\dots,j_k$. Since X is an infinitesimal automorphism of P if and only if the matrix $(\partial\xi^i/\partial x^j)$ belongs to the Lie algebra $\mathfrak{g}$ of G, we may conclude that X is an infinitesimal automorphism of P if and only if, for each fixed $j_2,\dots,j_k$, the matrix $(a^i_{j_1 j_2\dots j_k})_{i,j_1=1,\dots,n}$ belongs to the Lie algebra $\mathfrak{g}$. This motivates the following definition.

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(n; \mathbf{R})$. For $k=0, 1, 2, \ldots$, let $\mathfrak{g}_k$ be the space of symmetric multilinear mappings

$$t\colon \underbrace{\mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{(k+1)\text{-times}} \to \mathbf{R}^n$$

such that, for each fixed $v_1, \ldots, v_k \in \mathbf{R}^n$, the linear transformation

$$v \in \mathbf{R}^n \to t(v, v_1, \ldots, v_k) \in \mathbf{R}^n$$

belongs to $\mathfrak{g}$. In particular, $\mathfrak{g}_0 = \mathfrak{g}$. We call $\mathfrak{g}_k$ the *k-th prolongation* of $\mathfrak{g}$. The first integer k such that $\mathfrak{g}_k = 0$ is called the *order* of $\mathfrak{g}$. If $\mathfrak{g}_k = 0$, then $\mathfrak{g}_{k+1} = \mathfrak{g}_{k+2} = \cdots = 0$. If $\mathfrak{g}_k \neq 0$ for all k, then $\mathfrak{g}$ is said to be of *infinite type.*

Proposition 1.4. *A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n; \mathbf{R})$ is of infinite type if it contains a matrix of rank 1 as an element.*

Proof. Let e be a nonzero element of $\mathbf{R}^n$ and α a nonzero element of the dual space of $\mathbf{R}^n$. Then the linear transformation defined by

$$v \in \mathbf{R}^n \to \langle \alpha, v \rangle \, e \in \mathbf{R}^n$$

is of rank 1, and conversely, every linear transformation of rank 1 is given as above. Assume that the transformation above belongs to $\mathfrak{g}$. For each positive integer k, we define

$$t(v_0, v_1, \ldots, v_k) = \langle \alpha, v_0 \rangle \langle \alpha, v_1 \rangle \cdots \langle \alpha, v_k \rangle \, e, \qquad v_i \in \mathbf{R}^n.$$

Then t is a nonzero element of $\mathfrak{g}_k$. q.e.d.

We say that a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n; \mathbf{R})$ is *elliptic* if it contains no matrix of rank 1. Proposition 1.4 means that if $\mathfrak{g}$ is of finite order, then it is elliptic.

Each Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n; \mathbf{R})$ gives rise to a graded Lie algebra $\sum_{k=-1}^{\infty} \mathfrak{g}_k$, where $\mathfrak{g}_{-1} = \mathbf{R}^n$. The bracket of $t \in \mathfrak{g}_p$ and $t' \in \mathfrak{g}_q$ is defined by

$$\begin{aligned}[t, t'](v_0, v_1, \ldots, v_{p+q}) = {} & \frac{1}{p!\,(q+1)!} \sum t\big(t'(v_{j_0}, \ldots, v_{j_q}), v_{j_{q+1}}, \ldots, v_{j_{p+q}}\big) \\ & - \frac{1}{(p+1)!\,q!} \sum t'\big(t(v_{k_0}, \ldots, v_{k_p}), v_{k_{p+1}}, \ldots, v_{k_{p+q}}\big).\end{aligned}$$

In particular, if $t \in \mathfrak{g}_p$, $p \geqq 0$, and $v \in \mathfrak{g}_{-1} = \mathbf{R}^n$, then

$$[t, v](v_1, \ldots, v_p) = t(v, v_1, \ldots, v_p).$$

We explicitly set $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$. This definition is motivated by the following geometrical consideration. Suppose $t = (a^i_{j_0 \ldots j_p}) \in \mathfrak{g}_p$ and $t' = (b^l_{k_0 \ldots k_q}) \in \mathfrak{g}_q$ in terms of components and consider the corresponding vector fields:

$$X=\frac{1}{(p+1)!}\sum a^i_{j_0\dots j_p}x^{j_0}\dots x^{j_p}\frac{\partial}{\partial x^i},$$

$$Y=\frac{1}{(q+1)!}\sum b^i_{k_0\dots k_q}x^{k_0}\dots x^{k_q}\frac{\partial}{\partial x^i}.$$

Then $[X, Y]$ corresponds to $[t, t']$. Thus, the graded Lie algebra $\sum_{k=-1}^{\infty}\mathfrak{g}_k$ may be considered as the Lie algebra of infinitesimal automorphisms $X=\sum \xi^i\frac{\partial}{\partial x^i}$ with polynomial components ξ^i of the flat G-structure $P=\mathbf{R}^n\times G$ on $\mathbf{R}^n$.

For a survey on G-structures, see expository articles of Chern [1], [2]; the latter contains an extensive list of publications on the subject. See also Sternberg's book [1], A. Fujimoto [2], [3], Bernard [1].

The group of automorphisms of a compact elliptic structure or a G-structure of finite type will be shown to be a Lie transformation group (see §§ 4 and 5, respectively). These two cases cover a substantial number of interesting geometric structures whose automorphism groups are Lie groups. By considering G-structures of higher degree, we can bring such structures as projective structures under this general scheme (see § 8 of this chapter and Chapter IV). The group of automorphisms of a bounded domain or a similar complex manifold is also a Lie group (see § 1 of Chapter III), but this does not come under the general scheme. This book does not touch area-measure structures (Brickell [1]), nor pseudo-conformal structures of real hypersurfaces in complex manifolds (Morimoto-Nagano [1], Tanaka [3]) although automorphism groups of these structures are usually Lie groups.

2. Examples of G-Structures

Example 2.1. $G=\mathrm{GL}(n;\mathbf{R})$ and $\mathfrak{g}=\mathfrak{gl}(n;\mathbf{R})$. The Lie algebra $\mathfrak{g}$ contains a matrix of rank 1 and is of infinite type. A G-structure on M is nothing but the bundle $L(M)$ of linear frames and is obviously integrable. Every diffeomorphism of M onto itself is an automorphism of this G-structure and every vector field on M is an infinitesimal automorphism.

Example 2.2. $G=\mathrm{GL}^+(n;\mathbf{R})$ and $\mathfrak{g}=\mathfrak{gl}(n;\mathbf{R})$, where $\mathrm{GL}^+(n;\mathbf{R})$ means the group of matrices with positive determinant. The Lie algebra $\mathfrak{g}$ is of infinite type. A manifold M admits a $\mathrm{GL}^+(n;\mathbf{R})$-structure if and only if it is orientable; this is more or less the definition of orientability. A $\mathrm{GL}^+(n;\mathbf{R})$-structure on M may be considered as an orientation of M and is obviously integrable. A diffeomorphism of M onto itself is an automorphism of a $GL^+(n; \mathbf{R})$-structure if and only if it is orientation preserving. Every vector field on M is an automorphism since every one-parameter group of transformations is orientation preserving.

Example 2.3. $G = SL(n; \mathbf{R})$ and $\mathfrak{g} = \mathfrak{sl}(n; \mathbf{R})$. Again, $\mathfrak{g}$ contains a matrix of rank 1 and is of infinite type. The natural action of $GL(n; \mathbf{R})$ on $\mathbf{R}^n$ induces an action of $GL(n; \mathbf{R})$ on $\Lambda^n \mathbf{R}^n$ such that

$$Av = \det(A) \cdot v \quad \text{for } A \in GL(n; \mathbf{R}) \quad \text{and } v \in \Lambda^n \mathbf{R}^n.$$

The group $GL(n; \mathbf{R})$ is transitive on $\Lambda^n \mathbf{R}^n - \{0\}$ with isotropy subgroup $SL(n; \mathbf{R})$ so that $\Lambda^n \mathbf{R}^n - \{0\} = GL(n; \mathbf{R})/SL(n; \mathbf{R})$. It follows that the cross sections of the bundle $L(M)/SL(n; \mathbf{R})$ are in one-to-one correspondence with the *volume elements* of M, i.e., the n-forms on M which vanish nowhere. In other words, an $SL(n; \mathbf{R})$-structure is nothing but a volume element on M. It is clear that M admits an $SL(n; \mathbf{R})$-structure if and only if it is orientable. We claim that every $SL(n; \mathbf{R})$-structure is integrable. Indeed, let U be a coordinate neighborhood with local coordinate system $x^1, \ldots, x^n$ and let $\varphi = f\,dx^1 \wedge \cdots \wedge dx^n$ be the volume element corresponding to the given $SL(n; \mathbf{R})$-structure. Let $y^1 = y^1(x^1, \ldots, x^n)$ be a function such that $\partial y^1/\partial x^1 = f$. Then

$$\varphi = f\,dx^1 \wedge \cdots \wedge dx^n = dy^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

which shows that the coordinate system $y^1, x^2, \ldots, x^n$ is admissible with respect to the given $SL(n; \mathbf{R})$-structure. A diffeomorphism of M onto itself is an automorphism of the $SL(n; \mathbf{R})$-structure if and only if it preserves the volume element φ. Let X be a vector field on M. The function δX defined by

$$L_X \varphi = (\delta X) \cdot \varphi$$

is called the *divergence* of X with respect to φ. Clearly, X is an infinitesimal automorphism of the $SL(n; \mathbf{R})$-structure if and only if $\delta X = 0$. For $SL(n; \mathbf{R})$-structures, see §6.

Example 2.4. $G = GL(m; \mathbf{C})$ and $\mathfrak{g} = \mathfrak{gl}(m; \mathbf{C})$. We consider $GL(m; \mathbf{C})$ (resp. $\mathfrak{gl}(m; \mathbf{C})$) as a subgroup of $GL(2m; \mathbf{R})$ (resp. a subalgebra of $\mathfrak{gl}(2m; \mathbf{R})$) in a natural manner, i.e.,

$$A_1 + iA_2 \in GL(m; \mathbf{C}) \to \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} \in GL(2m; \mathbf{R})$$
$$\text{or} \quad \mathfrak{gl}(m; \mathbf{C}) \qquad\qquad \text{or} \quad \mathfrak{gl}(2m; \mathbf{R}).$$

Let $z^1, \ldots, z^m$ be the natural coordinate system in $\mathbf{C}^m$ and $z^j = x^j + i x^{m+j}$, $j = 1, \ldots, m$. Then the identification $\mathbf{C}^m = \mathbf{R}^{2m}$ given by

$$(z^1, \ldots, z^m) \to (x^1, \ldots, x^{2m})$$

induces the preceding injections

$$\mathrm{GL}(m;\mathbf{C})\to\mathrm{GL}(2m;\mathbf{R}) \quad \text{and} \quad \mathfrak{gl}(m;\mathbf{C})\to\mathfrak{gl}(2m;\mathbf{R}).$$

The multiplication by i in $\mathbf{C}^m$, i.e.,

$$(z^1,\ldots,z^m)\to(iz^1,\ldots,iz^m),$$

induces a linear transformation

$$(x^1,\ldots,x^m,x^{m+1},\ldots,x^{2m})\to(-x^{m+1},\ldots,-x^{2m},x^1,\ldots,x^m)$$

of $\mathbf{R}^{2m}$, which will be denoted by J. Since $i^2=-1$, we have $\mathsf{J}^2=-I$. In matrix form,

$$\mathsf{J}=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}.$$

The group $\mathrm{GL}(m;\mathbf{C})$ (resp. the algebra $\mathfrak{gl}(m;\mathbf{C})$), considered as a subgroup of $\mathrm{GL}(2m;\mathbf{R})$ (resp. a subalgebra of $\mathfrak{gl}(2m;\mathbf{R})$), is given by

$$\mathrm{GL}(m;\mathbf{C})=\{A\in\mathrm{GL}(2m;\mathbf{R});\ A\mathsf{J}=\mathsf{J}A\}$$
$$\mathfrak{gl}(m;\mathbf{C})=\{A\in\mathfrak{gl}(2m;\mathbf{R});\ A\mathsf{J}=\mathsf{J}A\}.$$

Since $\mathfrak{g}_k$ consists of all symmetric multilinear mappings of $\mathbf{C}^m\times\cdots\times\mathbf{C}^m$ ($k+1$ times) into $\mathbf{C}^m$, the Lie algebra $\mathfrak{g}$ is of infinite type. Every element of $\mathfrak{g}$, considered as an element of $\mathfrak{gl}(2m;\mathbf{R})$ is of even rank. Hence, $\mathfrak{g}$ is elliptic. The $\mathrm{GL}(m;\mathbf{C})$-structure on a manifold M (of dimension $2m$) are in one-to-one correspondence with the tensor field J of type (1, 1) on M such that

$$J_x\circ J_x=-I_x \qquad (\text{or simply, } J\circ J=-I),$$

where J_x is the endomorphism of the tangent space $T_x(M)$ given by J and I_x is the identity transformation of $T_x(M)$. The correspondence is given as follows. Given a tensor field J with $J\circ J=-I$, we consider, at each point x of M, only those linear frames $u\colon \mathbf{R}^{2m}\to T_x(M)$ satisfying $u\circ\mathsf{J}=J_x\circ u$. The subbundle of $L(M)$ thus obtained is the corresponding $\mathrm{GL}(m;\mathbf{C})$-structure on M. A tensor field J with $J\circ J=-I$ or the corresponding $\mathrm{GL}(m;\mathbf{C})$-structure is called an *almost complex structure.* We claim that *an almost complex structure is integrable (as a* $\mathrm{GL}(m;\mathbf{C})$*-structure) if and only if it comes from a complex structure.* (Before we explain this statement, we should perhaps remark an almost complex structure J is often called integrable if a certain tensor field of type (1, 2), called the torsion or Nijenhuis tensor, vanishes.) It is a deep result of Newlander and Nirenberg [1] that the two definitions coincide. For the real analytic case, see, for instance, Kobayashi-Nomizu [1, vol. 2; p. 321]. The theorem of Newlander-Nirenberg is equivalent to the statement that an $\mathrm{GL}(m;\mathbf{C})$-structure is integrable if and only if it admits a torsionfree affine connection (see Fröhlicher [1]). Let M be a complex manifold of complex dimension m with local coordinate system $z^1, \dots, z^m$ where $z^j = x^j + i y^j$. We have the natural almost complex structure J on M defined by

$$J(\partial/\partial x^j) = \partial/\partial y^j \qquad j = 1, \dots, m,$$
$$J(\partial/\partial y^j) = -\partial/\partial x^j \quad j = 1, \dots, m.$$

The almost complex structure J thus obtained is integrable since

$$(\partial/\partial x^1, \dots, \partial/\partial x^m, \partial/\partial y^1, \dots, \partial/\partial y^m)$$

gives a local cross section of the $\mathrm{GL}(m; \mathbf{C})$-structure defined by J. Conversely, if an almost complex structure J is integrable as a $\mathrm{GL}(m; \mathbf{C})$-structure and if $x^1, \dots, x^{2m}$ is an admissible local coordinate system, then $J(\partial/\partial x^j) = \partial/\partial x^{m+j}$ and $J(\partial/\partial x^{m+j}) = -\partial/\partial x^j$ for $j = 1, \dots, m$. If we set $z^j = x^j + i x^{m+j}$, then the complex coordinate system $z^1, \dots, z^m$ turns M into a complex manifold. A diffeomorphism f of M onto itself is an automorphism of an almost complex structure J if and only if $f_* \circ J = J \circ f_*$, where $f_*\colon T(M) \to T(M)$ is the differential of f. If J is integrable, an automorphism f is nothing but a holomorphic diffeomorphism. A vector field X on M is an infinitesimal automorphism of an almost complex structure J if and only if

$$[X, JY] = J([X, Y]) \qquad \text{for all vector field } Y \text{ on } M.$$

For further properties of an almost complex structure, see Kobayashi-Nomizu [1; Chapter IX].

Example 2.5. $G = \mathrm{O}(n)$ and $\mathfrak{g} = \mathfrak{o}(n)$. The Lie algebra $\mathfrak{g}$ is of order 1. Let $t \in \mathfrak{g}_1$ and (t^i_{jk}) the components of t. By definition, $t^i_{jk} = t^i_{kj}$. Since $\mathfrak{o}(n)$ consists of skew-symmetric matrices, we have $t^i_{jk} = -t^j_{ik}$. Hence,

$$t^i_{jk} = -t^j_{ik} = -t^j_{ki} = t^k_{ji} = t^k_{ij} = -t^i_{kj} = -t^i_{jk},$$

thus proving $t^i_{jk} = 0$. To each Riemannian metric on M, there corresponds the bundle of orthonormal frames over M. This gives a one-to-one correspondence between the Riemannian metrics on M and the $\mathrm{O}(n)$-structures on M. An $\mathrm{O}(n)$-structure is integrable if and only if the corresponding Riemannian metric is flat, i.e., it has vanishing curvature. An automorphism of an $\mathrm{O}(n)$-structure is an isometry of the corresponding Riemannian metric. An infinitesimal automorphism of an $\mathrm{O}(n)$-structure is an infinitesimal isometry or Killing vector field. We shall discuss isometries and Killing vector fields in detail later (see Chapter II).

More generally, let $G = \mathrm{O}(p, q)$, $n = p + q$, be the orthogonal group defined by a quadratic form $u_1^2 + \dots + u_p^2 - u_{p+1}^2 - \dots - u_n^2$. Then $\mathfrak{o}(p, q)$ is also of order 1. There is a natural one-to-one correspondence between the pseudo-Riemannian metrics of signature q on M and the $O(p,q)$-structures on M. An $O(p,q)$-structure is integrable if and only if the corresponding pseudo-Riemannian metric has vanishing curvature. It should be remarked that, although every paracompact manifold admits a Riemannian metric, it may not in general admit a pseudo-Riemannian metric of signature q for $q \neq 0, n$. For automorphism of pseudo-Riemannian manifolds, see Tanno [1, 2].

Example 2.6. $G = CO(n)$ and $\mathfrak{g} = \mathfrak{co}(n)$, $n \geqq 3$. By definition,

$$CO(n) = \{A \in GL(n; \mathbf{R});\ {}^tAA = cI,\ c \in \mathbf{R},\ c > 0\},$$

$$\mathfrak{co}(n) = \{A \in \mathfrak{gl}(n; \mathbf{R});\ {}^tA + A = cI,\ c \in \mathbf{R}\}.$$

Thus, $CO(n) = O(n) \times \mathbf{R}^+$ and $\mathfrak{co}(n) = \mathfrak{o}(n) + \mathbf{R}$, where $\mathbf{R}^+$ denotes the multiplicative group of positive real numbers. The Lie algebra $\mathfrak{co}(n)$ is of order 2 and the first prolongation $\mathfrak{g}_1$ is naturally isomorphic to the dual space $\mathbf{R}^{n*}$ of $\mathbf{R}^n$. To determine $\mathfrak{g}_1$, let $t = (t^i_{jk})$ be an element of $\mathfrak{g}_1$. Since the kernel of the homomorphism $A \in \mathfrak{co}(n) \to \mathrm{trace}(A) \in \mathbf{R}$ is precisely $\mathfrak{o}(n)$ and since $\mathfrak{o}(n)$ is of order 1, the linear mapping

$$t = (t^i_{jk}) \in \mathfrak{g}_1 \to \xi = \left(\frac{1}{n} \sum_i t^i_{ik}\right) \in \mathbf{R}^{n*}$$

is injective. The kernel is the first prolongation of $\mathfrak{o}(n)$. (The factor of $\frac{1}{n}$ is, of course, not important). To see that this mapping is also surjective, we have only to observe that $\xi = (\xi_k)$ is the image of t with $t^i_{jk} = \delta^i_j \xi_k + \delta^i_k \xi_j - \delta^j_k \xi_i$. To prove $\mathfrak{g}_2 = 0$, let $t = (t^h_{ijk}) \in \mathfrak{g}_2$. For each fixed k, t^h_{ijk} may be considered as the components of an element in $\mathfrak{g}_1$ and hence can be uniquely written

$$t^h_{ijk} = \delta^h_i \xi_{jk} + \delta^h_j \xi_{ik} - \delta^i_j \xi_{hk}.$$

Since t^h_{ijk} must be symmetric in all lower indices, we have $\sum_h t^h_{hjk} = \sum_h t^h_{hkj}$, from which follows $\xi_{jk} = \xi_{kj}$. From $\sum_h t^h_{hjk} = \sum_h t^h_{jkh}$, we obtain $(n-2)\xi_{jk} = -\delta_{jk} \cdot \sum_h \xi_{hh}$, from which follows $(n-2)\sum_h \xi_{hh} = -n \sum_h \xi_{hh}$ and, hence, $\sum_h \xi_{hh} = 0$. From $(n-2)\xi_{jk} = -\delta_{jk} \cdot \sum_h \xi_{hh} = 0$ and $n \geqq 3$, we conclude $\xi_{jk} = 0$. (The reader who prefers an index-free proof is referred to Kobayashi-Nagano [3, III; p. 686].) A $CO(n)$-structure is called a *conformal structure*. We say that two Riemannian metrics on M are *conformally equivalent* if one is a multiple of the other by a positive function. The conformal equivalence classes of Riemannian metrics on M are in a natural one-to-one correspondence with the $CO(n)$-structure on M. A conformal structure is integrable if and only if any Riemannian metric corresponding to the structure is locally conformally equivalent to $(dx^1)^2+\cdots+(dx^n)^2$ with respect to a suitable local coordinate system $x^1, \ldots, x^n$. Thus, a conformal structure is integrable if and only if it is conformally flat in the classical sense (see Eisenhart [1]). Consequently, the integrability of a conformal structure is equivalent to the vanishing of the so-called conformal curvature tensor of Weyl (provided $n \geqq 3$). Given a Riemannian metric g on M, a diffeomorphism f of M onto itself (resp. a vector field X on M) is a conformal transformation, i.e., an automorphism of the conformal structure (resp. an infinitesimal conformal transformation, i.e., an infinitesimal automorphism of the conformal structure) if and only if

$$f^* g = \rho \cdot g \qquad (\text{resp. } L_X g = \sigma \cdot g),$$

where ρ (resp. σ) is a positive function (resp. a function) on M. Conformal structures and their automorphisms will be discussed in Chapter IV.

The reason we excluded the case $n=2$ is that $\mathrm{CO}(2)$ (resp. $\mathfrak{co}(2)$) is naturally isomorphic to $\mathrm{GL}(1; \mathbf{C})$ (resp. $\mathfrak{gl}(1; \mathbf{C})$). For this reason, the conformal differential geometry in dimension 2 differs significantly from that in higher dimensions. In particular, we note that every $\mathrm{CO}(2)$-structure, i.e., $\mathrm{GL}(1; \mathbf{C})$-structure is integrable; this is nothing but the existence of isothermal coordinate systems.

The results for $\mathrm{CO}(n)$-structures can be easily generalized to $\mathrm{CO}(p, q)$-structures, where $\mathrm{CO}(p, q) = \mathrm{O}(p, q) \times \mathbf{R}^+$ is defined by a quadratic form of signature q.

Example 2.7. $G = U(m)$ and $\mathfrak{g} = \mathfrak{u}(m)$. Since $\mathfrak{u}(m)$ is a subalgebra of $\mathfrak{o}(2m)$ which is of order 1 (cf. Examples 2.4 and 2.5), it is also of order 1. A $U(m)$-structure on a $2m$-dimensional manifold M is called an *almost hermitian structure;* it consists of an almost complex structure and a hermitian metric. Since $U(m) = \mathrm{GL}(m; \mathbf{C}) \cap \mathrm{O}(2m)$, a $U(m)$-structure may be considered as an intersection of a $\mathrm{GL}(m; \mathbf{C})$-structure and an $\mathrm{O}(2m)$-structure. A $U(m)$-structure is integrable if and only if the underlying almost complex structure is integrable (so that M is a complex manifold) and the hermitian metric has vanishing torsion and curvature. A diffeomorphism of M onto itself is an automorphism of a $U(m)$-structure if and only if it is an automorphism of the underlying $\mathrm{GL}(m; \mathbf{C})$- and $\mathrm{O}(2m)$-structures. Similarly, for an infinitesimal automorphism. For automorphisms of hermitian manifolds, see Tanno [3].

Example 2.8. $G = \mathrm{Sp}(m; \mathbf{R})$ and $\mathfrak{g} = \mathfrak{sp}(m; \mathbf{R})$. We recall that $\mathrm{Sp}(m; \mathbf{R})$ is the group of linear transformations of $\mathbf{R}^{2m}$ leaving the form

$$u^1 \wedge u^{m+1} + \cdots + u^m \wedge u^{2m}$$

invariant, where $u^1, \ldots, u^{2m}$ is the natural coordinate system in $\mathbf{R}^{2m}$. In other words,

$$\mathrm{Sp}(m; \mathbf{R}) = \{A \in \mathrm{GL}(2m; \mathbf{R});\ {}^tA\mathrm{J}A = \mathrm{J}\},$$

$$\mathfrak{sp}(m; \mathbf{R}) = \{A \in \mathfrak{gl}(2m; \mathbf{R});\ {}^tA\mathrm{J} + \mathrm{J}A = 0\},$$

where

$$\mathrm{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Since $\mathfrak{sp}(m; \mathbf{R})$ consists of matrices of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & {}^tA_1 \end{pmatrix} \quad \text{with } {}^tA_2 = A_2 \quad \text{and} \quad {}^tA_3 = A_3,$$

it contains an element of rank 1 and, hence, is of infinite type. The $\mathrm{Sp}(m, \mathbf{R})$-structures on a $2m$-dimensional manifold M are in a natural one-to-one correspondence with the 2-forms ω on M of maximum rank (i.e., $\omega^m \neq 0$ everywhere).

Since both $\mathrm{GL}(m; \mathbf{C})$ and $\mathrm{Sp}(m; \mathbf{R})$ contain $U(m)$ as a maximal compact subgroup, a manifold M admits an $\mathrm{Sp}(m; \mathbf{R})$-structure if and only if it admits a $\mathrm{GL}(m; \mathbf{C})$-structure. An $\mathrm{Sp}(m; \mathbf{R})$-structure is called an *almost symplectic structure* or an *almost Hamiltonian structure*. If an almost symplectic structure is integrable with admissible coordinate system $x^1, \ldots, x^{2m}$ so that

$$\omega = dx^1 \wedge dx^{m+1} + \cdots + dx^m \wedge dx^{2m},$$

then $d\omega = 0$. Conversely (see Appendix 1), if the form ω defining an almost symplectic structure is closed, then $\omega = dx^1 \wedge dx^{m+1} + \cdots + dx^m \wedge dx^{2m}$ for a suitable local coordinate system $x^1, \ldots, x^{2m}$ and the structure is integrable. An integrable almost symplectic structure is called a *symplectic structure* or a *Hamiltonian structure*. We observe that if an almost symplectic structure admits a torsionfree affine connection, then it is integrable. For the 2-form ω defining an almost symplectic structure is parallel with respect to such a connection and hence is closed. (In calculating $d\omega$ in terms of a local coordinate system, partial differentiation may be replaced by covariant differentiation when the connection is torsionfree, see for instance Kobayashi-Nomizu [1, vol. 1; p. 149]). A diffeomorphism f of M onto itself is an automorphism of the symplectic structure defined by a 2-form ω if and only if $f^*\omega = \omega$. Similarly, X is an infinitesimal automorphism if and only if $L_X\omega = 0$. An (infinitesimal) automorphism of a symplectic structure is called an *(infinitesimal) symplectic transformation*.

Set

$$\mathrm{CSp}(m; \mathbf{R}) = \{A \in \mathrm{GL}(2m; \mathbf{R});\ {}^tA\mathrm{J}A = c\mathrm{J},\ c \in \mathbf{R}^+\} = \mathrm{Sp}(m; \mathbf{R}) \times \mathbf{R}^+,$$

$$\mathfrak{csp}(m; \mathbf{R}) = \{A \in \mathfrak{gl}(2m; \mathbf{R});\ {}^tA\mathrm{J} + \mathrm{J}A = c\mathrm{J},\ c \in \mathbf{R}\} = \mathfrak{sp}(m; \mathbf{R}) + \mathbf{R}.$$