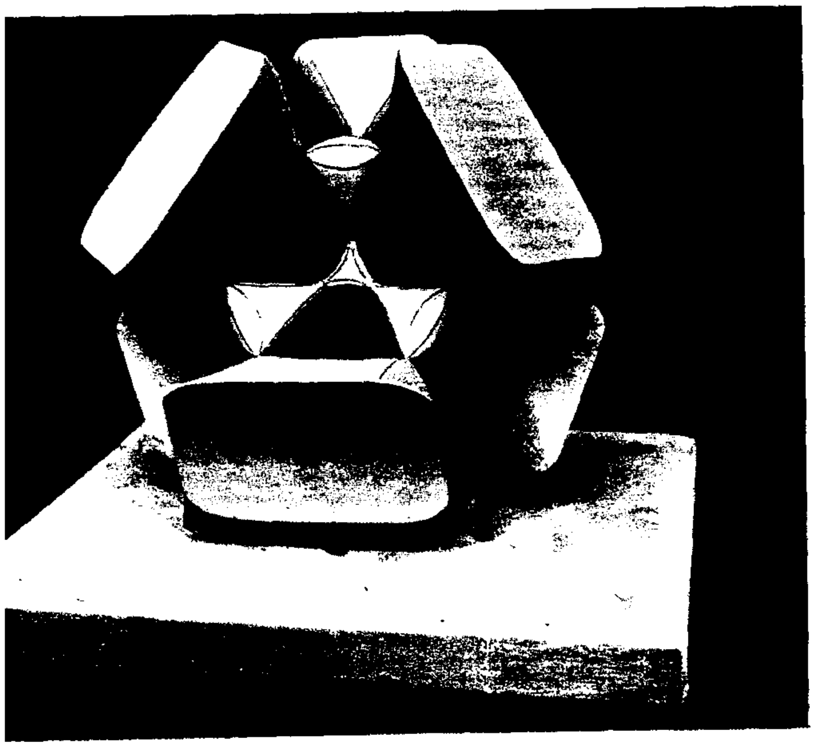

MER'S QUARTIC SURFACE

BY

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CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York   Port Chester

Melbourne   Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1905
Reissued as a paperback with a Foreword by W. Barth
in the Cambridge Mathematical Library series 1990

Printed in Great Britain by J. W. Arrowsmith Ltd, Bristol

ISBN 0 521 39790 1 paperback

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FOREWORD

In 1916 there appeared, a decade after Hudson's book, Jessop's book on quartic surfaces (Jessop, 1916). In his preface Jessop states 'The admirable work written by the late R. W. H. T. Hudson, entitled *Kummer's Quartic Surface*, renders unnecessary the inclusion of this subject', i.e. the inclusion of Kummer surfaces in his own book. Now, almost a century later, and after so many deep changes of methods, style and aims in geometry, Hudson's work stands as admirably as when it impressed the experienced author, Jessop.

The theory of Kummer surfaces includes so many beautiful aspects of geometry that A. Weil used the first letter of its inventor's name, when he introduced the notion $K3$ for the class of surfaces to which Kummer surfaces belong. To some extent the beauty and importance of the subject explains the attraction of Hudson's book. To a larger extent, however, the work impresses by its masterly presentation. Hudson starts at the beginning – nothing is assumed to be known except for the elements of geometry in space. Everything else is explained by the author. Of course this cannot mean he proves everything; often he borrows results from other sources. And naturally he does not always give exact definitions of his mathematical tools, because ideas like 'group actions' or 'varieties' were developed in full precision only later. But he does have the ability to explain!

One of the outcomes of Bourbaki's efforts to modernize our science is the experience, probably not wanted by Bourbaki, that explaining is not the same as defining and proving something. It is the art of presenting material attractively, vigorously, following a path that seems natural to a reader not yet familiar with the final aim. Hudson's book stands out above all others I know by the masterful way in which he, though then not yet thirty, presented the many-faceted material.

Today, the formal theory of surfaces has reached a certain stage of completeness. Of course there are important open questions. But it seems that major efforts concentrate on solving these concrete questions and not on developing further the formal theory. (The two aspects of mathematics are, admittedly, hard to separate.) Some of these questions are already touched on very directly, but elegantly, in Hudson's book: the classification of quartic surfaces, the description of moduli spaces for abelian surfaces with level structures, with distinguished subgroups, the automorphism group of a Kummer surface, ... So I believe a reissue of this book now, when the original is hard to get on the market, will be of use to many doing active research.

In 1972 I saw this book for the first time when checking through the geometry section of Leiden's Mathematics library – where books are arranged by fields and not by the cruel regime of the alphabet only. Since then it has become my favourite book in Mathematics. Several times I have tried to order a copy from catalogues of old books, but never was fast enough. So it was a great pleasure for me that Cambridge University Press offered me the unique opportunity of taking part in this reissue. The book speaks for itself. I want to describe below, in the language of today, the contents of its eighteen chapters, hoping that this will be of use to those readers who are educated in the modern way.

I. KUMMER'S CONFIGURATION

Kummer's configuration of sixteen points and phases in three-space is most easily understood after describing its group of symmetries. In some fixed coordinate system this group is generated by the operations of changing the sign of two coordinates and permuting the four coordinates in pairs (see § 4). It is one of the simplest examples of what is nowadays called the Heisenberg group (see Mumford (1966)). The configuration is obtained from a general plane (or dually a point) as the sixteen transforms of this plane. There will be an orbit of sixteen points under this group such that each configuration plane contains six of these points and, conversely, each point in the orbit lies on six planes: Kummer's 16_6, see § 3. The six points in any plane are in special position in the sense that they lie on a conic (§ 7).

When describing this geometry, Hudson simultaneously introduces the notion of a group. Since he refers to Burnside's book for the definition of a group, and since he obviously has problems when explaining the group operation, we may safely conclude that in about 1900 a group was some complicated concept, not known to the general mathematical public.

In § 1 Hudson describes a pencil of quartic surfaces which are invariant under the group just mentioned, even under some bigger group. This pencil contains three tetrahedrons as surfaces decomposing into four planes. On several later occasions these tetrahedrons appear again, but not the general surfaces in the pencil. This is an example for elementary geometry leading immediately to nontrivial surfaces. In fact these surfaces are birational to Kummer surfaces again. (They are related to self-products $E \times E$ of elliptic curves. The reader may consider this an exercise and use chapter II in Jessop (1916) as a hint.)

II. THE QUARTIC SURFACE

In this chapter Hudson develops the elementary geometry of quartic surfaces in three-space carrying sixteen nodes, which is due to Kummer. Beginning with the definition that a surface is singular where it 'ceases to be approximately flat' he repeats Kummer's proof that its nodes and tropes (= planes touching the surface along a conic) form a 16_6. In chapter I he stated – without convincing proof – that each 16_6 was of the form considered in this chapter.

Hudson proceeds to study the quartic equation of the surface by projecting it from one node. He introduces six parameters $k_1, \ldots, k_6$ for six points on a conic. The conic is the basis for the tangent cone at the fixed node, and the six points give the points of contact on this conic of the six tropes. He uses two copies of the square root

$$\sqrt{(u-k_1)(u-k_2)\ldots(u-k_6)}$$

to parametrize the surface. This is nothing but the uniformization of Kummer's surface by the symmetric product of a hyperelliptic (genus 2-)curve, thus by the jacobian of this curve. The numbers $k_1, \ldots, k_6$ are the coordinates of the six branch points of this curve. § 11 is quite interesting for modern readers. Up to here not one word has been used to say whether Hudson considers the complex surface or its real points only. Here he describes the 'shape' of the surface, of course the real one. He accidentally remarks that the equation of the surface is the linear combination of a tetrahedron and the square of a quadric. He then specializes to the most symmetric case – which, by the way, was exhibited 30 years earlier in a series of plaster models by Kummer – and tries to explain the shape of it. The beautifully symmetric drawings on pages 21 and 22 should be considered a challenge to each owner of a personal computer able to do graphics.

III. THE ORTHOGONAL MATRIX OF LINEAR FORMS

It is a deep satisfaction for us newly educated geometers to have a look at §§ 12–14. Hudson struggles with orthogonal matrices, or rather he tries to present them to his readers in a harmless way. We stop smiling, however, when we see what Hudson does with these orthogonal matrices: he obtains linear and quadratic relations between the sixteen trope planes of the Kummer surface and he explicitly represents sets of eight associated points amongst the sixteen nodes. (Such a set is the intersection of three quadrics.) The ten fundamental quadrics referred to in § 17 are the ten eigenfunctions of the Heisenberg group mentioned above for its operation on the (ten-dimensional) space of quadratic functions on projective three-space. Just as on this space of quadratic polynomials, the Heisenberg group acts commutatively on the six-dimensional space of alternating functions in the coordinates, i.e. on the five-space in which the Plücker quadric lies. Consequently there are six invariant linear forms in the Plücker coordinates, i.e. six invariant quadratic complexes described in § 18.

IV. LINE GEOMETRY

Line geometry is the geometry of lines in three-space, i.e. the geometry of the Plücker quadric. A linear (line-) complex is a hyperplane section of this Plücker quadric. It is defined by an alternating bilinear form on the four-dimensional vector space underlying projective three-space. Orthogonality with respect to this form leads to a symmetric relation between points and planes (null-point to a plane, null-plane for a point) and between lines (two lines are polar if their defining two-dimensional vector spaces are orthogonal to each other). Any first-year student following a course on Linear Algebra knows that, but has probably never heard about apolarity, a kind of commutativity of linear complexes, explained in absolutely elementary geometric terms in § 21.

Apolarity of two linear complexes just means their alternating bilinear forms wedge each other to zero (for these tensors a symmetric bilinear relation). Writing S, T for the correlation defined by these tensors, Hudson writes $ST = TS$ for the apolarity condition, where we would view the tensors S, T as anti-selfadjoint maps $\mathbb{C}^4 \to (\mathbb{C}^4)^*$ and write $S^{-1}T = T^{-1}S$ for apolarity. Then clearly $S^{-1}T$ is an involutory linear map on $\mathbb{C}^4$. Or we could say, two linear complexes are apolar if their hyperplanes in five-space are polar with respect to the Plücker quadric. Then, for dimensional reasons, there are maximally six mutually apolar linear complexes leading to fifteen involutions on our basic three-space. With great love for details, Hudson shows geometrically (see in particular § 26) that together with the identity they constitute the Heisenberg group again. As a by-product of this digression on line geometry (nowadays part of multilinear algebra) Hudson obtains a 16_6 configuration starting from one of its vertices by taking the six null planes for six apolar complexes and the ten polar planes for ten related quadrics. This is the most basic access to the fundamental $16 = 10 + 6$ decomposition of sixteen objects by parity.

Only in the last three sections of this chapter does Hudson use coordinates to complement his, up-to-then, purely geometric method.

This chapter illustrates how easy it is to approach beautiful geometry. Of course, the language is old-fashioned, but I have never seen a modern exposition of all this lovely geometry.

V. THE QUADRATIC COMPLEX AND CONGRUENCE

A quadratic complex is the variety of lines in three-space parametrized by a quadratic hypersurface in five-space (that projective space where the Plücker quadric lies). In general these two quadratic forms can be simultaneously diagonalized. The corresponding coordinate system defines six hyperplanes on five-space. They are mutually apolar linear complexes, so the situation from the preceding chapter returns.

It was Kummer who, studying quadratic line complexes as models for the system of light rays in an optical instrument, observed a particular surface of focal points for pencils of lines in the complex. This surface, called 'singular surface' in the classical literature and denoted by Φ, is the Kummer surface for the quadratic complex. It has a smooth model, a complete intersection in five-space of three quadrics: the Plücker quadric, the quadratic complex, and a third quadric intimately attached to both of these. The birational relation between this smooth $K3$-surface in five-space and its image with sixteen nodes in three-space is extremely rich in geometric properties, which, though not obvious, are accessible with some effort. A modern account is given in the last fifty pages of the voluminous monograph by Griffiths & Harris (1978).

This discovery of Kummer is without doubt one of the most basic facts of algebraic geometry. It is the starting point of many modern theories.

Compared with any modern exposition, Hudson presents in the seventeen pages of this chapter the relation between the quadratic congruence and its Kummer surface with charming elegance. Of course, he uses extensively his geometric preparations. But most notably he does not bear the burden of the theory of our time.

VI. PLÜCKER'S COMPLEX SURFACE

Plücker's surface is a degenerate form of Kummer's surface when two eigenvalues are coincident, as we would say. What can go wrong if one tries to diagonalize simultaneously two quadrics in five-space, and what happens to the Kummer surface then is presented with great care (e.g. in the table on pp. 230–232 in Jessop (1969)). There are fifty-five cases.

Plücker's surface is a rational quartic with a double line. Its equation and other properties form the contents of this short chapter, at the end of which we again find a section on the shape of the surface. I cannot claim to understand the beautiful picture there. But I am convinced it is worth while to spend some computer time on it.

VII. SETS OF NODES

This is probably the chapter for which a modern presentation would differ most from Hudson's text. Not before chapter XVI does Hudson touch on the uniformization of Kummer's surface by theta functions, i.e. the fact that the Kummer surface is a quotient of an abelian surface. The sixteen nodes, corresponding to the sixteen half-periods on the abelian surface, can be parametrized by the sixteen points in affine four-space over the tiny field $\mathbb{F}_2$ with two elements. The symmetries discussed in this chapter - not without notational difficulty - are governed by the geometry of quadrics over this field $\mathbb{F}_2$. It is fascinating to see, however, how purely combinatorial properties of this sixteen-point set translate into geometry in three-space.

The distinction between Rosenhain and Goepel tetrads is nothing but the distinction between isotropic and non-isotropic planes in this affine space over the finite field.

VIII. EQUATIONS OF KUMMER'S SURFACE

This chapter gives two different quartic equations for Kummer's surface in three-space. These two equations differ by the coordinate system in which they are written, i.e. by the coordinate tetrahedron.

In the first case (§ 53), a coordinate system is used in which the symmetries of the Heisenberg group have the particularly simple form described in chapter I. In modern language this is the choice of a level structure on the 2-torsion group of the covering abelian surface; the parameters A, B, C, D used in the equation are the moduli for abelian surfaces with level-two structure, and the cubic identity satisfied by them is the equation for the moduli space of abelian surfaces with such a level structure.

In the second case a Rosenhain tetrahedron is used. This is the choice in the 2-torsion group $\mathbb{F}_2^4$ of some non-isotropic affine subspace $\mathbb{F}_2^4$. The parameters u, v, w, s (the latter of weight 2) are the moduli of such structures on abelian surfaces. This gives an explicit identification of this moduli space with a weighted projective space.

The explicit description of moduli spaces for abelian varieties, as in these two classical cases, is at the moment a field of quite active research.

IX. SPECIAL FORMS OF KUMMER'S SURFACE

In this chapter geometric properties of special Kummer surfaces are described.

Characterizing the surface by the six numbers $k_1, \ldots, k_6$, the cases are distinguished according to how many involutions leave this sixtuplet invariant, i.e. according to the automorphism group of the hyperelliptic curve: a tetrahedroid is the Kummer surface for the jacobian of a curve admitting an involution other than the hyperelliptic one. Multiple tetrahedroids belong to hyperelliptic curves with even bigger automorphism groups. The list of five cases altogether should be compared with the now well-known classification of hyperelliptic curves according to their group of automorphisms (Geyer, 1974).

X. THE WAVE SURFACE

The wave surface, also called Fresnel's wave surface, is related to the expansion of light rays in certain crystals; for a mathematical discussion see e.g. Knörrer (1986). Up to projective transformations it is the same as the tetrahedroid introduced in chapter IX. However, because of its connection with the physical world, its metrical properties are of particular interest. They are studied in this chapter. Also a parametrization of the surface in terms of two elliptic functions is given. This is possible because the additional involution on the hyperelliptic curve is elliptic, i.e. it comes from a map onto an elliptic curve. Consequently the jacobian surface is isogeneous to a product of two elliptic curves.

XI. REALITY AND TOPOLOGY

Reality problems are some of the most intriguing of algebraic geometry. On the one hand, they are very natural; algebraic geometry started in the 'real world', and when visualizing this kind of geometry, reality is of obvious importance. On the other hand, they are either totally trivial, or extremely complicated. Modern theory only has a modest hold on curves and surfaces in three-space.

A real Kummer surface is one given by an equation with real coefficients. Such a surface need not have real points at all, nor need the sixteen nodes be real.

This chapter classifies all real Kummer surfaces according to their number of real nodes, their number of 'sheets' or 'pieces' into which the real part of the surface decomposes (a notion used intuitively, and probably not too easy to make precise), and according to the elliptic or hyperbolic curvature of these sheets.

XII. GEOMETRY OF FOUR DIMENSIONS

For today's mathematician this chapter is the strangest in the book. The author thought it necessary at this point to introduce his readers to elementary projective geometry in four dimensions. Later on he defines a variety to be a hypersurface in four-space.

In between, he shows how to obtain Kummer's 16_6 configuration by the operations of intersection and projection from the figure of six points in four-space in general position. He applies this to the quartic hypersurface which is the dual of the Segre cubic primal. The latter in symmetric form is given by

$$x_1+\cdots+x_6=0, \qquad x_1^3+\cdots+x_6^3=0.$$

He shows that a hyperplane section of the quartic has fifteen nodes, hence if it is the intersection with a tangent plane it has sixteen nodes and is a Kummer surface. Dually, when projecting the Segre cubic onto three-space from one of its points, the 'enveloping cone is the cone over' such a Kummer surface. In modern terminology: the branch surface is Kummer. Now these results, obtained in a quite elementary way and clearly with a minimum of theory, are of considerable interest in present-day algebraic geometry: phrased somewhat more precisely, they mean that the dual quartic to the Segre primal is a moduli space for abelian surfaces with level-two structure. Each point on this threefold is the modulus point for the abelian cover of its Kummer tangent section. A modern treatment putting this in proper perspective is given, for example, in v. d. Geer (1982).

XIII. ALGEBRAIC CURVES ON THE SURFACE

XIV. CURVES OF DIFFERENT ORDERS

These two chapters give general facts about algebraic curves on the surface and apply them to get something like a classification. Again the arguments given here are elementary, and the resulting equations explicit. Anybody in need of such equations will be very grateful to Hudson. (Such need could arise, for example, if one wanted to draw a computer picture of the surface and of some curves on it.) However, the presentation is really very far from a proper understanding. Modern methods of dealing with symmetric divisors on an abelian surface are so much more powerful. So these two chapters seem to me a part of the book which, from our point of view, should not be placed here, but only after the author shows how to parametrize the surface by theta functions in chapter XVI.

Of course, the advantage of this treatment is that it avoids transcendent theory, so to a large extent is probably independent of the characteristic of the base field.

XV. WEDDLE'S SURFACE

Weddle's surface is another model of a Kummer surface as space quartic, however with only six nodes. It is obtained from the usual model by the linear system of cubics passing through ten 'even' nodes. Hudson first introduces his reader to the theory of birational transformations and linear systems with base points. He touches on the question of which other quartic surfaces are birational images of Kummer surfaces. Weddle's surface is the first interesting case here. He gives geometric properties of the surface and derives its equation.

Again, nowadays we would not try to deal with such questions without the theory of the Picard group and Picard lattice of a Kummer surface. But we should not be conceited: the most natural question in this field, namely to describe the automorphism group of a Kummer surface (= the birational maps into itself) at present is unsolved. We know only a little more than did the classical geometers (see § 120, the last one in this book).

XVI. THETA FUNCTIONS

XVII. APPLICATIONS OF ABEL'S THEOREM

Now, finally, these long-awaited functions are introduced. In the usual, somewhat frightening, formalism, theta-relations are proved, the Kummer surface is parametrized, and the transcendental theory is sketched. Abel's theorem about abelian integrals and divisors of rational functions is applied to derive geometric properties of the Kummer surface. For example, several kinds of tetrahedra inscribed into and circumscribed about the surface are constructed.

These chapters contain a wealth of beautiful geometry. With our modern insight, they are not very astonishing. However, I do not know any other place in the literature where they are collected so nicely.

XVIII. SINGULAR KUMMER SURFACES

Here the classical phrase 'singular' is used to describe a Kummer surface, if the Picard number of its abelian cover is bigger than one. This can happen, for example, if the abelian cover is isogeneous to a product of elliptic curves or, more generally, if it admits a non-trivial endomorphism. It is characterized by relations (linear and quadratic) between the periods of the abelian cover. The surfaces in the moduli space defined by such relations (Humbert surfaces) are at present still of great interest. For example, Hilbert modular surfaces belong to them.

First Hudson describes elliptic Kummer surfaces, i.e. those for which the abelian cover is isogeneous to a product of elliptic curves, and specifies curves on them. For example, he recovers the tetrahedroid there.

Surfaces with 'invariant' five and eight are considered too. Humbert's beautiful descriptions of these surfaces as double covers of the plane ramified over special hexagons circumscribed about a conic are obtained. The abelian cover of a surface with invariant five has real multiplication in the field $\mathbb{Q}\sqrt{5}$, and for invariant eight we have real multiplication in the field $\mathbb{Q}\sqrt{2}$.

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REFERENCES

Geyer, W. D. (1974). *Invarianten binärer Formen*, Springer LNM, 36–39.

Griffiths, Ph. and Harris, J. (1978). *Principles of Algebraic Geometry*, Wiley-Interscience.

Knörrer, H. (1986). *Die Fresnelsche Wellenfläche*, Mathematische Miniaturen 3, Birkhäuser.

Jessop, C. M. (1969). *A Treatise on the Line Complex*, Cambridge (1903), reprinted by Chelsea.

Jessop, C. M. (1916). *Quartic Surface with Singular Points*, Cambridge.

Mumford, D. (1966). On the equations defining abelian varieties. I. *Inv. math.* **1**, 287–354.

v. d. Geer, G. (1982). On the geometry of a Siegel modular threefold. *Math. Ann.* **260**, 317–350.