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PREFACE

The momentum of the mind is all toward abstraction.

- Wallace Stevens, Opus Posthumous

Within the framework of twistor theory the structure of spacetime is relegated, in contrast to the position which it has held since the beginning of the twentieth century, to a status of secondary character. Whereas in the past spacetime has always served as the background against which phenomena are to be interpreted—and indeed, according to Einstein's theory of gravitation, spacetime serves moreover as a basic dynamical entity itself—the new view which the twistor theorists are advocating takes twistor space, with the many rich and variegated aspects of its complex analytic structure, as the primary descriptive device and dynamical construction in terms of which phenomena are to be understood.

The difficulties inherent in a spacetime description have long been appreciated by many authors. Julian Schwinger, for example, in his preface to Selected Papers on Quantum Electrodynamics summarizes the situation aptly when he remarks that "... The localization of charge with indefinite precision requires for its realization a coupling with the electromagnetic field that can obtain arbitrarily large magnitudes. The resulting appearance of divergences, and contradictions, serves to deny the basic measurement hypothesis. We conclude that a convergent theory cannot be formulated consistently within the framework of present space-time concepts. To limit the magnitude of interactions while retaining the customary coordinate description is contradictory, since no mechanism is provided for precisely localized measurements." With a similar attitude towards this question Einstein, at the end of The Meaning of Relativity, concludes that "One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality." Of course when he refers to a continuum Einstein means spacetime, taken with its usual real differentiable structure. In twistor theory, however, the continuum which arises is that of the complex number system, and those aspects of the geometry of twistor space which are of interest to physics stem more specifically from its complex analytic structure, rather than its real differentiable structure. The general characterization of the structures which can arise in the case of complex analytic manifolds has been the subject of intense investigation by mathematicians, especially with the advent of the powerful techniques of sheaf cohomology theory. One of the precepts of twistor theory is that here, within a suitably formulated sheaf cohomological framework, we have the proper basis for a "purely algebraic" description that is compatible both with the ideas of relativity and with the principles of quantum mechanics.

This view has met with a reasonable degree of success, and it has been possible, using methods of algebraic geometry and complex analytic geometry, for twistor theorists to assemble the outlines of a new approach to elementary particle physics. The subject is still in its infancy and in a rapid state of development, and thus many of its results are only of a preliminary character and are both subject to and deserving of considerable modification and improvement. In spite of their tentative nature, it seemed appropriate nonetheless to prepare an account of some of these matters for a wider audience, with the hope that it might stimulate or otherwise prove a useful aid in further and more extensive research into the subject. With this purpose in mind the following study is presented.

Although a fair amount of background material is covered in Chapters 2 and 3, the reader previously uninitiated into the mysteries of twistor theory may find it necessary to consult some additional references. For the two-component spinor formalism see Pirani (1965), Penrose (1968a), and the forthcoming book by Rindler and Penrose. For further reading in basic twistor theory see Penrose (1967), Penrose and MacCallum (1972), and Penrose (1975a). Although a specialized knowledge of elementary particle physics is not necessary, at the outset, for reading this volume, it is assumed nonetheless that the reader is familiar with basic quantum mechanics, and is acquainted already, to some extent, with the quark model.

The author is indebted to many of his colleagues for their help in the preparation and development of this material, particularly to R. Penrose who originated many of the ideas discussed here, and who has acted as a constant source of illumination and inspiration. G.A.J. Sparling has contributed extensively to this work, and the author wishes to thank him for many helpful discussions. I would also like to thank many of my colleagues at Oxford and elsewhere, including D.M. Blasius, M. Eastwood, M.L. Ginsberg, A. Hodges, S.A. Huggett, T.R. Hurd, R. Jozsa, E.T. Newman, A. Popovich, Z. Perjés, I. Robinson, M. Sheppard, L. Smarr, P. Sommers, K.P. Tod, Tsou S.T., M. Walker, R.S. Ward, and N.M.J. Woodhouse, for useful conversations and suggestions related to the work described herein. The author is grateful to B.S. DeWitt, C.M. DeWitt, R. Matzner, L. Shepley, H.J. Smith, and the late Alfred Schild, as well as other colleagues at the University of Texas at Austin, for their hospitality shown during the author's 1974 visit, when some of the ideas preliminary to the material described here were worked out. The author has profited much from his regular visits, supported by the Clark Foundation, to the University of Texas at Dallas, and he would like to thank I. Ozsvath, W. Rindler, I. Robinson, and J.R. Robinson for their hospitality. Likewise the author has benefited from his visits to the Astronomy Department at the University of Virginia, and gratitude is expressed to W. Saslaw, and other colleagues there, for their hospitality. I am grateful to J. Ehlers, M.L. Ginsberg, C.J. Isham, R. Penrose, G.A.J. Sparling, and N.M.J. Woodhouse for reading earlier drafts of the manuscript and contributing many corrections and helpful suggestions for improvement.

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This volume is dedicated to my mother and my father.

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CHAPTER 1

INTRODUCTORY REMARKS

Progress in any aspect is a movement
through changes in terminology.

- Wallace Stevens, Opus Posthumous

This study will touch on a variety of topics concerning twistor theory and elementary particle physics. A few of these topics will be treated in some detail, but none exhaustively. The purpose of this work is to describe how it is possible, using twistor methods, to gain some understanding of the microscopic structural degrees of freedom responsible for the properties of elementary particles.

In a very general sense the methodology of twistor theory consists simply of the application of techniques of complex analytic geometry to problems in physics. Inherent in the twistor program are many changes in terminology, whereby a number of the familiar concepts of physics are reexpressed in the language of algebraic geometry and analytic geometry. "The physicist always prefers to sacrifice the less perfect concepts of physics rather than the simpler, more perfect, and more lasting concepts of geometry, which form the solidest foundation of all his theories", said Mach, and there is certainly a good deal of reason in his remark: but the twistor philosophy goes one step further, and insists that within geometry itself one can discover all the laws of physics.

The organization of this volume is as follows. Chapters 2 and 3 view twistor space from the standpoint of classical physics. Algebraic geometry is to complex analytic geometry as classical physics is to quantum physics—and in Chapters 2 and 3 twistor space is explored with various tools of algebraic geometry. Most of the information in Chapter 2 is standard background material, and is summarized here for the reader previously unacquainted with twistor theory. Twistors are first defined in terms of classical systems of zero rest mass—that is to say, classical special relativistic systems defined by a null momentum and an angular momentum which is related to the momentum in such a way that twistors transform in a natural way under the action of the group SU(2,2), and, in particular, the Poincaré group. In §2.4 it is shown that twistors can be characterized in terms of the solutions of a certain differential equation called the "twistor equation". In §§2.5 and 2.6 twistors are described in terms of the geometry of complex projective 3-space P^3. Complex projective lines in P^3 correspond to points in complex Minkowski space; using this correspondence (the "Klein representation") various aspects of the geometry of spacetime are expressed in twistor terms, and vice-versa.

In Chapter 3 it is shown how massive systems can be built up out of two or more twistors. The momentum and the angular momentum are described in terms of a single two-index symmetric "kinematical twistor". Theorem 3.3.1 shows how any massive system can be decomposed into two or more twistor constituents. Thus, massive systems (at the classical level) can always be regarded as being "made up" out of twistors. Twistors are, in a certain sense, the elementary constituents of matter. For a given momentum and angular momentum there are internal degrees of freedom which yet remain, mixing the various twistor constituents. Theorems 3.4.2 and 3.4.14 show the relevant groups which leave the momentum and angular momentum of an n-twistor system invariant. These groups are called the "n-twistor internal symmetry groups", and, for each value of n, contain U(n) as a subgroup. It is proposed that these internal degrees of freedom are in some sense responsible for the phenomenological unitary groups which arise naturally in elementary particle classification schemes (e.g., SU(3)). In §3.5 a center of mass twistor is introduced for n-twistor systems. This construction plays a useful role in a number of problems.

In Chapter 4 the rules of twistor quantization are introduced for systems composed of a single twistor. It is shown how solutions of the zero rest mass equations can be obtained in terms of holomorphic functions defined over suitable domains of twistor space. Both positive and negative helicity fields are discussed, and the differences in the relevant contour integral formulae for evaluating the fields, in the two cases, are noted. The positive frequency condition is discussed in §4.5, and the whole procedure is illustrated with the example of an elementary state.

In Chapter 5 massive fields are described in terms of holomorphic functions of two or more twistors. It is proposed that observables correspond to holomorphic differential operators with polynomial coefficients. Explicit expressions are presented for the operators corresponding to momentum, angular momentum, mass, and spin. In §5.5 the operators corresponding to "internal" observables are discussed, and are described explicitly in the cases of one, two, and three twistors.

In Chapter 6 the scheme is applied to the low-lying baryons—that is to say, the N(949) octet and the Δ(1232) decimet. After a brief review of the quark model (described in a language suitable for our purposes) it is demonstrated how the low-lying baryons can be represented in terms of certain types of holomorphic functions of three twistors. Baryons are not regarded as bound states of quarks. No color degrees of freedom are introduced.

In Chapter 7 the methods of Chapter 6 are extended so as to apply to more general systems. Mesons are introduced as quark-antiquark bound states, described in terms of holomorphic functions of six twistors. The charge conjugation quantum number plays a crucial role in the representation of these states. Orbital angular momentum is described in twistor terms, and it is shown how orbital excitations of the quark-antiquark system lead to meson resonances. Baryon resonances are represented as excitations of a quark-diquark bound state. The deuteron is briefly discussed, from a twistor point of view, in the last section of Chapter 7. In Chapter 8, after a review of the properties of leptons and of parity violation in weak interactions, a model for sequential leptons is built up in twistor terms. Chapters 9 and 10 are concerned with further mathematical developments in the theory. In Chapter 9 the methods of sheaf cohomology are introduced, and these are applied to various problems in Chapter 10, the aim being to sharpen up much of the material of the previous chapters, and to open up the doors to more extensive developments.

The tentative nature of any general inferences that can now be put forward in connection with the twistor particle program, or, for that matter, twistor theory in general, should undoubtedly be apparent to anybody working in this subject. One need merely consider the vast range of phenomena which so far have resisted any formulation in twistor terms whatsoever. Nonetheless, significant conclusions are being drawn along certain lines, and are receiving continually increasing support.

In particular, the central role of the twistor program in connection with Einstein's theory seems to me now firmly established, and there does not seem to be any reason now why particle physics as a whole should not be amenable to treatment within the framework of twistor theory.

CHAPTER 2

ASPECTS OF THE GEOMETRY OF TWISTOR SPACE

2.1 Classical Systems of Zero Rest Mass.

There are various ways of building up the framework of twistor theory, and it must be said that it is not exactly clear where to begin. For the purposes of investigations into elementary particle physics a convenient, if not totally adequate, place to start is with the observation that a point Z^{α} in twistor space ($\alpha = 0,1,2,3$) can be represented naturally in terms of physical quantities as a classical system of zero rest mass.

Such a system is characterized by its total momentum P^a, which is null and future-pointing, and its angular momentum M^{ab} ($= -M^{ba}$) with respect to a particular choice of origin in spacetime.

Together these quantities must satisfy a relation to the effect that if we form the spin-vector

$$S_a = \frac{1}{2} \varepsilon_{abcd} P^b M^{cd} \tag{2.1.1}$$

then the proportionality $S_a = sP_a$ holds for some value of the number s. The magnitude of s is the spin of the system, and s itself is called the helicity. Positive helicity systems are called right-handed, and negative helicity systems are called left-handed.

There is a certain algebraic characterization of the momentum and angular momentum that ensures that together they constitute a zero rest mass (henceforth abbreviated ZRM) system:

2.1.2 Proposition. A pair $\{P^a , M^{ab}\}$ represents a ZRM system if and only if there exists a pair of spinors $(\omega^A , \pi_{A'})$ such that

$$P^a = \bar{\pi}^A \pi^{A'} \tag{2.1.3}$$

and

$$M^{ab} = i\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - i\bar{\omega}^{(A'}\pi^{B')}\varepsilon^{AB} , \tag{2.1.4}$$

where $\bar{\pi}^A$ is the complex conjugate of $\pi^{A'}$, and $\bar{\omega}^{A'}$ is the complex conjugate of ω^A.

Proof[1]. The existence of a spinor $\pi^{A'}$ such that equation (2.1.3) is satisfied is precisely the condition that P^a should be null and future-pointing.

The spin-relation $S^a = sP^a$ can be written $*M^{ab}P_b = sP^a$ where $*M^{ab} := \frac{1}{2}\varepsilon^{abcd}M_{cd}$ is the dual of M^{ab}. If we write

$$(2.1.5) \qquad *M^{ab} = -i\mu^{AB}\varepsilon^{A'B'} + i\bar{\mu}^{A'B'}\varepsilon^{AB} \quad ,$$

where μ^{AB} is a symmetric spinor, then the spin-relation, using equation (2.1.3), reads

$$(2.1.6) \qquad -i\mu^{AB}\bar{\pi}_B\pi^{A'} + i\bar{\mu}^{A'B'}\pi_{B'}\bar{\pi}^A = s\bar{\pi}^A\pi^{A'} \quad .$$

Contracting this relation with $\bar{\pi}_A$ yields $\mu^{AB}\bar{\pi}_A\bar{\pi}_B = 0$, which implies $-i\mu^{AB} = \omega^{(A}\bar{\pi}^{B)}$ for some choice of ω^A, the factor of $-i$ being included for later convenience. Finally, using the fact that equation (2.1.5) implies

$$(2.1.7) \qquad M^{ab} = \mu^{AB}\varepsilon^{A'B'} + \bar{\mu}^{A'B'}\varepsilon^{AB} \quad ,$$

we deduce equation (2.1.4). □

The spinor pair $(\omega^A , \pi_{A'})$ completely determines the ZRM system, and defines a point Z^α in twistor space according to the scheme

$$(2.1.8) \qquad (Z^0 , Z^1 , Z^2 , Z^3) = (\omega^0 , \omega^1 , \pi_{0'} , \pi_{1'}) \quad .$$

Note, on the other hand, that a ZRM system determines its associated twistor only up to an overall phase factor, since the momentum and the angular momentum are invariant under the transformation

$$(2.1.9) \qquad (\omega^A , \pi_{A'}) \longrightarrow e^{i\theta}(\omega^A , \pi_{A'}) \quad .$$

It is interesting to observe that the helicity of a ZRM system can be expressed directly in twistor terms. For this purpose it is useful to define the complex conjugate twistor $\bar{Z}_\alpha$ by the spinor pair $(\bar{\pi}_A , \bar{\omega}^{A'})$. A short calculation establishes that the inner product defined by

$$(2.1.10) \qquad Z^\alpha\bar{Z}_\alpha = \omega^A\bar{\pi}_A + \pi_{A'}\bar{\omega}^{A'}$$

is precisely twice the helicity of the system, i.e. we have $Z^{\alpha}\bar{Z}_{\alpha} = 2s$.

One might be inclined initially to think that the freedom expressed in (2.1.9) is of an irrelevant nature, and arises perhaps on account of some slight inadequacy in the representation that has been chosen for twistors in terms of systems of zero rest mass. Nothing could be further from the truth, however. One of the remarkable things about twistors is that they do, in fact, carry more information in them than just momentum and angular momentum. This fact takes on great significance, as we shall see, when quantum mechanics is brought into the picture.

2.2 The Action of the Poincaré Group.

It is of considerable interest to know how the action of the Poincaré group is expressed in twistor terms. Since our ultimate goal is to express various field quantities in terms of twistors, and since these field quantities must themselves be subject to a particular behavior under the action of the Poincaré group, it is of significance to study the action of the Poincaré group on twistors first.

Under the spacetime translation $x^a \longrightarrow x^a + r^a$ the angular momentum M^{ab} transforms according to the rule

(2.2.1) $$M^{ab} \longrightarrow M^{ab} + 2r^{[a}P^{b]} \quad .$$

It is not difficult to check that for a ZRM system the transformation on Z^{α} that induces (2.2.1) is

(2.2.2) $$\omega^{A} \longrightarrow \omega^{A} + ir^{AA'}\pi_{A'} \quad , \quad \pi_{A'} \longrightarrow \pi_{A'} \quad .$$

This transformation can therefore be regarded as defining the action of a spacetime translation on Z^{α}.

The action of a restricted Lorentz transformation on a ZRM system is specified by

(2.2.3) $$P_a \longrightarrow \Lambda_a{}^{b}P_b \quad , \quad M_{ab} \longrightarrow \Lambda_a{}^{c}\Lambda_b{}^{d}M_{cd} \quad .$$

For a restricted Lorentz transformation $\Lambda_a{}^{b}$ has the form

(2.2.4) $$\Lambda_a{}^{b} = \ell_A{}^{B}\bar{\ell}_{A'}{}^{B'} \quad ,$$

where $\ell_A{}^B$ is an element of the group SL(2,C), i.e. subject to the relation

$$\ell_A{}^C \ell_B{}^D \varepsilon_{CD} = \varepsilon_{AB} \quad . \tag{2.2.5}$$

The action on Z^α which induces (2.2.3) is easily verified to be:

$$\omega^A \longrightarrow -\ell^A{}_B \omega^B \quad , \quad \pi_{A'} \longrightarrow \bar{\ell}_{A'}{}^{B'} \pi_{B'} \quad . \tag{2.2.6}$$

By following a Lorentz transformation with a translation, we can realize the complete action of the restricted Poincaré group on a twistor. This can be conveniently expressed in the form

$$Z^\alpha \longrightarrow P^\alpha{}_\beta Z^\beta \quad , \tag{2.2.7}$$

where the transformation matrix $P^\alpha{}_\beta$ is given by

$$P^\alpha{}_\beta = \begin{pmatrix} -\ell^A{}_B & ir^{AA'}\bar{\ell}_{A'}{}^{B'} \\ 0 & \bar{\ell}_{A'}{}^{B'} \end{pmatrix} \quad , \tag{2.2.8}$$

with the usual laws of matrix multiplication applying in the contraction of $P^\alpha{}_\beta$ with the spinor parts of Z^β. That is to say, we have

$$\omega^A \longrightarrow -\ell^A{}_B \omega^B + ir^{AA'}\bar{\ell}_{A'}{}^{B'} \pi_{B'} \qquad \pi_{A'} \longrightarrow \bar{\ell}_{A'}{}^{B'} \pi_{B'} \tag{2.2.9}$$

for the spinor parts of equation (2.2.7).

2.3 The Group SU(2,2).

The complex conjugate twistor $\bar{Z}_\alpha$ undergoes the complex conjugate transformation $\bar{Z}_\alpha \longrightarrow \bar{P}_\alpha{}^\beta \bar{Z}_\beta$ when Z^α undergoes transformation (2.2.7). Since the helicity s is Poincaré invariant, the requirement that the inner product $Z^\alpha \bar{Z}_\alpha$ be preserved implies that $P^\gamma{}_\beta \bar{P}_\gamma{}^\alpha = \delta^\alpha_\beta$, where δ^α_β is the twistor Kronecker delta, given in spinor parts by:

$$\delta^\alpha_\beta = \begin{pmatrix} -\varepsilon^A{}_B & 0 \\ 0 & \varepsilon_{A'}{}^{B'} \end{pmatrix} \quad . \tag{2.3.1}$$

The set of all matrices $U^\alpha{}_\beta$ satisfying $U^\gamma{}_\beta \bar{U}_\gamma{}^\alpha = \delta^\alpha_\beta$ forms the group U(2,2). This can be seen as follows. Such transformation matrices preserve the norm $Z^{\alpha}\bar{Z}_{\alpha}$, which is given explicitly by

(2.3.2) $$Z^{\alpha}\bar{Z}_{\alpha} = \omega^{A}\bar{\pi}_{A} + \pi_{A'}\bar{\omega}^{A'} = \omega^{0}\bar{\pi}_{0} + \omega^{1}\bar{\pi}_{1} + \pi_{0'}\bar{\omega}^{0'} + \pi_{1'}\bar{\omega}^{1'} .$$

If new variables are introduced according to the scheme

(2.3.3) $$\omega^{0} = (w+y) \qquad \omega^{1} = (x+z) \qquad \pi_{0'} = (w-y) \qquad \pi_{1'} = (x-z)$$

where w, x, y, and z are complex, then

(2.3.4) $$\tfrac{1}{2}Z^{\alpha}\bar{Z}_{\alpha} = w\bar{w} + x\bar{x} - y\bar{y} - z\bar{z} \quad ,$$

which shows that the helicity is a quadratic Hermitian form of signature {++--}. The group U(2,2) is by definition the multiplicative group of complex linear transformations which preserve a quadratic Hermitian form of that signature.

The group SU(2,2) is the subgroup of U(2,2) consisting of matrices which, in addition to satisfying $U^{\gamma}{}_{\beta}\bar{U}_{\gamma}{}^{\alpha} = \delta^{\alpha}_{\beta}$, also preserve the twistor epsilon tensor $\varepsilon^{\alpha\beta\gamma\delta}$, i.e.:

(2.3.5) $$U^{\alpha}{}_{\xi}U^{\beta}{}_{\eta}U^{\gamma}{}_{\zeta}U^{\delta}{}_{\theta}\, \varepsilon^{\xi\eta\zeta\theta} = \varepsilon^{\alpha\beta\gamma\delta} \quad .$$

Condition (2.3.5) amounts to the same thing as requiring that $U^{\alpha}{}_{\beta}$ have unit determinant.

SU(2,2) is of special importance to physics inasmuch as it is locally isomorphic with the 15-parameter conformal group of compactified Minkowski space[(2)]. The restricted Poincaré group is a subgroup of SU(2,2). A description of the relationship between the two groups can be facilitated with the introduction of the so-called "infinity twistors", given by

(2.3.6) $$I^{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix} \qquad I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}$$

which, according to a scheme to be elaborated in Section 2.6, represent the vertex of the null cone at infinity.

The infinity twistors are skew-symmetric, are complex conjugates of one-another, and satisfy the tollowing relations:

$$(2.3.7) \qquad I^{\alpha\beta} I_{\beta\gamma} = 0 \quad , \quad I^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta} \quad , \quad I_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} I^{\gamma\delta} \quad .$$

Poincaré transformations are SU(2,2) transformations which have the property that they preserve the infinity twistors.

2.4 The Twistor Equation.

Another way in which twistor space arises naturally is as the solution set of the differential equation

$$(2.4.1) \qquad \nabla^{A'(A}\xi^{B)} = 0 \quad ,$$

which, accordingly, is sometimes called the twistor equation.

2.4.2 Proposition. The general solution of equation (2.4.1) is

$$(2.4.3) \qquad \xi^{A}(x) = \omega^{A} - ix^{AA'}\pi_{A'} \quad ,$$

where ω^A and $\pi_{A'}$ are constant.

Proof. Equation (2.4.1) can be written in the form

$$(2.4.4) \qquad \nabla^{B'B}\xi^{C} = \frac{1}{2}\varepsilon^{BC}\nabla^{B'}_{D}\xi^{D} \quad .$$

Taking a derivative, we have

$$(2.4.5) \qquad \nabla^{A'A}\nabla^{B'B}\xi^{C} = \frac{1}{2}\varepsilon^{BC}\nabla^{AA'}\nabla^{B'}_{D}\xi^{D} \quad ,$$

which, using $\nabla^{A'(A}\xi^{C)} = 0$, implies

$$(2.4.6) \qquad \varepsilon^{B(C}\nabla^{A)A'}\nabla^{B'}_{D}\xi^{D} = 0 \quad ,$$

showing that $\nabla^{B'}_{D}\xi^{D}$ is a constant spinor, which will be denoted $2i\pi^{B'}$, the factor of 2i being for convenience. Substituting this result back into equation (2.4.4), integration then gives (2.4.3), with ω^A appearing as a constant of integration. □

The pair $(\omega^A , \pi_{A'})$ defines the twistor Z^α , and $\xi^A(x)$ is called the associated spinor field(3) of the twistor Z^α. It can be checked that the natural action of the

Poincaré group on $\xi^A(x)$ agrees with the action on Z^α defined in Section 2.2.

2.5 α-Planes and β-Planes.

The location of a twistor Z^α in complex Minkowski space can be defined as the region for which the associated spinor field $\xi^A(x)$ vanishes. From (2.4.3) this is evidently the condition that

$$\omega^A = ix^{AA'}\pi_{A'} \quad . \tag{2.5.1}$$

Since equation (2.5.1) is linear in $x^{AA'}$, and represents a pair of conditions that these coordinates must satisfy, the solution for fixed ω^A and $\pi_{A'}$ must be a 2-plane. Moreover it should be obvious that if $x_0^{AA'}$ represents any particular point satisfying (2.5.1), then the general point satisfying this relation is $x_0^{AA'} + \lambda^A\pi^{A'}$, where the spinor λ^A is arbitrary. So the location of the twistor Z^α is the 2-plane consisting of all the endpoint positions of a complex vector $\lambda^A\pi^{A'}$ springing from the point $x_0^{AA'}$. Each such complex vector is null. Moreover, since $\pi^{A'}$ is fixed, each such vector is orthogonal to any other. Thus Z^α corresponds to a null 2-plane in Minkowski space.

A point W_α in <u>dual</u> twistor space is represented by a spinor pair $(\sigma_A, \tau^{A'})$. Associated with W_α is a solution of the "primed" twistor equation

$$\nabla^{A(A'}\eta^{B')} = 0 \tag{2.5.2}$$

given by

$$\eta^{A'} = \tau^{A'} + ix^{A'A}\sigma_A \quad . \tag{2.5.3}$$

By analogy with Proposition (2.4.2) it is not difficult to see that equation (2.5.3) gives the general solution of (2.5.2). The locus of the dual twistor W_α is given by

$$\tau^{A'} = -ix^{A'A}\sigma_A \quad , \tag{2.5.4}$$

the region where $\eta^{A'}$ vanishes. In this case if $x_0^{A'A}$ represents any particular solution to (2.5.4) then the general solution is given by $x_0^{A'A} + \lambda^{A'}\sigma^A$.

It is of interest to note that in complex Minkowski space there are two distinct systems of null 2-planes. The so-called α-planes are those null 2-planes which correspond to twistors of valence $[{}^1_0]$, i.e. the Z^α-type twistors. The β-planes are those null 2-planes which correspond to twistors of valence $[{}^0_1]$, i.e. the W_α-type twistors.

Any two distinct α-planes have a unique intersection point in complex Minkowski space. If the corresponding twistors are denoted Z_1^α and Z_2^α then for an intersection point one must solve simultaneously the algebraic equations,

$$\omega_1^A = ix^{AA'}\pi_{1A'}, \qquad \omega_2^A = ix^{AA'}\pi_{2A'} \quad . \tag{2.5.5}$$

Assuming that $\pi_{1A'}$ is not proportional to $\pi_{2A'}$, the unique solution to these equations is given by the formula

$$ix^{AA'} = (\omega_1^A\pi_2^{A'} - \omega_2^A\pi_1^{A'})/(\pi_{1A'}\pi_2^{A'}) \quad , \tag{2.5.6}$$

as can readily be checked.

In manifestly twistorial terms the solution for the intersection point can be represented by the skew product of the two twistors. In particular, if we put

$$X^{\alpha\beta} = (Z_1^\alpha Z_2^\beta - Z_2^\alpha Z_1^\beta)/(Z_1^\alpha Z_2^\beta I_{\alpha\beta}) \tag{2.5.7}$$

where a normalization factor has been included so as to ensure that $X^{\alpha\beta}I_{\alpha\beta} = 2$, then one finds

$$X^{\alpha\beta} = \begin{bmatrix} -\frac{1}{2}x_d x^d \varepsilon^{AB} & ix^A{}_{B'} \\ -ix_{A'}{}^B & \varepsilon_{A'B'} \end{bmatrix} \tag{2.5.8}$$

for the spinor parts of $X^{\alpha\beta}$, after a short calculation. Thus a spacetime point $x^{AA'}$ is represented in twistor terms by a simple skew-symmetric twistor $X^{\alpha\beta}$ (recall that "simple" here means that $X^{[\alpha\beta}X^{\gamma]\delta} = 0$) satisfying the normalization condition $X^{\alpha\beta}I_{\alpha\beta} = 2$, where $I_{\alpha\beta}$ is the infinity twistor defined in (2.3.6).

The dual description of the same spacetime point is formed by taking $X_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}X^{\gamma\delta}$. The complex conjugate spacetime point $\bar{x}^a$ is described dually by the complex conjugate twistor $\bar{X}_{\alpha\beta}$. The condition that a spacetime point should be real is that the dual twistor should be equal to the complex conjugate twistor, i.e. $X_{\alpha\beta} = \bar{X}_{\alpha\beta}$.

If $X^{\alpha\beta}$ and $Y^{\alpha\beta}$ represent, according to the description given above, the spacetime points x^a and y^a , respectively, then the quantity

$$-X^{\alpha\beta}Y_{\alpha\beta} = (x^a-y^a)(x_a-y_a) \tag{2.5.9}$$

is the norm of the spacetime separation of the two points. In particular, if $x^a = v^a-iw^a$ where v^a and w^a are real, then

$$\frac{1}{4} X^{\alpha\beta}\bar{X}_{\alpha\beta} = w_a w^a \tag{2.5.10}$$

is the norm of the imaginary part of x^a , which is a Poincaré invariant quantity.

That region of complex Minkowski space (CM) for which w^a is timelike and future-pointing will be called (notwithstanding some apparently unavoidable terminological awkwardness) the future - tube, and will be denoted CM^+. The region for which w^a is timelike and past-pointing will be denoted CM^-.

2.6 Projective Twistor Space.

An α-plane does not determine a twistor uniquely, but rather, as should be evident from equation (2.5.1), only up to an overall scale factor. An equivalence class of twistors all of which are proportional to each other is called a projective twistor, and by projective twistor space (PT) we mean the set of all such equivalence classes.

It is clear that projectively a twistor does not have a well-defined norm. Nevertheless, projectively the sign of the norm still makes sense, and thus PT can be divided into three parts denoted PT^+, PN, and PT^- according as to whether the norm $Z^\alpha\bar{Z}_\alpha$ is positive, zero, or negative.

Often we will use the twistor coordinates Z^α to denote the associated equivalence class in projective twistor space. In that case we refer to the components of Z^α as the homogeneous coordinates for the corresponding point in PT. The systematic use of homogeneous coordinates has the marvelous catalytic effect of simplifying much of the