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Arthur L. Besse

Manifolds all of whose Geodesics are Closed

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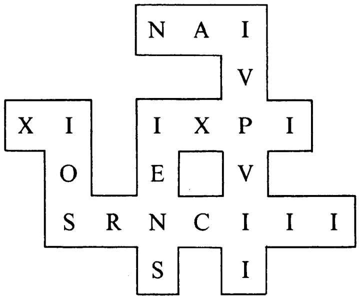


Cher lecteur,

J'entre bien tard dans la sphère étroite des écrivains au double alphabet, moi qui, il y a plus de quarante ans déjà, avais accueilli sur mes terres un général épris de mathématiques. Il m'avait parlé de ses projets grandioses en promettant d'ailleurs de m'envoyer ses ouvrages de géométrie.

Je suis entiché de géométrie et c'est d'elle dont je voudrais vous parler, oh! certes pas de toute la géométrie, mais de celle que fait l'artisan qui taille, burine, amène, gauchit, peaufine les formes. Mon intérêt pour le problème dont je veux vous entretenir ici, je le dois à un ami ébéniste.

En effet comme je rendais un jour visite à cet ami, je le trouvai dans son atelier affairé à un tour. Il se retourna bientôt, puis, rayonnant, me tendit une sorte de toupie et me dit: «Monsieur Besse, vous qui calculez les formes avec vos grimoires, que pensez-vous de ceci?» Je le regardai interloqué. Il poursuivit: «Regardez! Si vous prenez ce collier de laine et si vous le maintenez fermement avec un doigt placé n'importe où sur la toupie, eh bien! la toupie passera toujours juste en son intérieur, sans laisser le moindre espace.» Je rentrai chez moi, fort étonné, car sa toupie était loin d'être une boule. Je me mis alors au travail...

Après quelques recherches, je compris que, sans être une boule, sa toupie était une surface de révolution dont toutes les géodésiques avaient (sensiblement) la même longueur. En consultant mes grimoires, comme disait mon ami l'ébéniste, je m'aperçus que Darboux avait déjà posé ce problème d'équations différentielles en 1894. En 1903 Zoll avait donné un exemple explicite d'une telle surface et Funk avait trouvé en 1913 une méthode pour en construire une famille continue (mais V. Guillemin n'a rendu cette construction rigoureuse que de nos jours). Gambier, quant à lui, a consacré de multiples études dans les années 20 à l'extension de ces constructions, en particulier au cas des polyèdres. Il prit aussi plaisir à questionner certain de mes amis, mais ceci est une autre histoire.

Tout au long de ces années, les mathématiciens travaillaient aussi à formaliser leur art... Depuis le début du siècle ils réussirent à abstraire les caractéristiques essentielles de la notion de surface plongée dans notre espace de tous les jours, créant les variétés différentielles, riemanniennes et autres choses savantes. Mieux, parmi les espaces riemanniens symétriques d'E. Cartan, une nouvelle classe d'espaces à géodésiques fermées de même longueur apparut: les espaces projectifs sur les nombres réels, sur les nombres complexes et sur les quaternions (ainsi que l'ésotérique plan projectif des octaves de Cayley).

Voilà un sujet comportant bien peu d'exemples, me direz-vous! C'est cela qui incita Blaschke à conjecturer que, sur les espaces projectifs, seule leur structure symétrique pouvait être à géodésiques toutes fermées de même longueur.

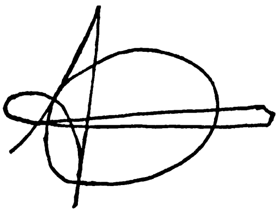
A propos de ce problème bien spécifique de géométrie, les techniques les plus variées des mathématiques ont eté mises en œuvre, d'où l'idée directrice de ce livre: rassembler tous les outils permettant une approche de ce problème géométrique. Aussi j'expose les résultats de G. Reeb (1950) repris en 1974 par A. Weinstein se fondant sur la géométrie symplectique; le théorème de R. Bott (1954) utilise la topologie algébrique, celui de L. Green (1963) la géométrie intégrale du fibré unitaire, ceux de R. Michel (1972) outre la géométrie riemannienne l'analyse géométrique, enfin ceux de J. Duistermaat et V. Guillemin (1975) l'analyse des distributions.

Mais, malgré tous ces résultats partiels, on ne peut pas dire que le problème ait beaucoup avancé. J'ai donc été très heureux de pouvoir accueillir en Mai 1975 dans mes terres, au pays que hantent toujours mouflons et cumulo-nimbus, une table ronde consacrée à l'étude des espaces à géodésiques fermées* et autour de laquelle la cuisine locale a été fort appréciée. C'est à cette occasion que mes amis m'ont convaincu qu'il fallait que je fasse un livre**. Mes réceptions du mardi après-midi ont vite pris un caractère animé. Pourtant je crus plusieurs fois que je n'arriverais pas à atteindre le but que je m'étais assigné.

Il m'a été possible d'ordonner ce que je connais de ce problème et de préciser les nombreuses questions non résolues qui s'y rattachent à l'occasion de nombreuses discussions avec entre autres: Pierre Bérard, Lionel Bérard Bergery, Marcel Berger, Jean-Pierre Bourguignon, Yves Colin de Verdière, Annie Deschamps, Jacques Lafontaine, René Michel, Pierrette Sentenac. Le texte de ce livre est trop formalisé à mon goût, mais vous savez comment sont devenus les mathématiciens! De plus, écrire dans la langue du poète W. Hamilton a été pour moi une dure contrainte. Je crains, cher lecteur, que vous ne pâtissiez des conséquences.

En espérant tout de même avoir pu vous faire partager mon goût pour la géométrie, je reste Géodésiquement

Vôtre,



Arthur BESSE

Le Faux, le 15 Juin 1976

* L'Université Paris VII et le C.N.R.S. ont rendu cette réunion possible et je les en remercie.

** Je remercie les Editions Springer d'avoir bien voulu accepter de le publier.

Table of Contents

Chapter 0. Introduction

A. Motivation and History

0.1. Suppose that a Riemannian manifold (M, g) is one of the following: $(S^d$, can), $(\mathbb{R}P^d$, can), $(\mathbb{C}P^n$, can), $(\mathbb{H}P^n$, can), $(\mathbb{C}aP^2$, can), namely one of the compact symmetric spaces of rank one, the so-called CROSSes in 3.16—endowed with their canonical Riemannian structure *denoted* by can. Then all geodesics of (M, g) are closed (if you doubt this fact see 3.31). For a mathematician the basic question is then: are these examples the only manifolds all of whose geodesics are closed (up to isometry, of course)? If not, give counter-examples and then try to classify—even describe—all manifolds with that property.

0.2. To be more precise let us introduce for a Riemannian manifold (M, g) the two following assertions:

SC: there exists a positive real number l such that for every unit tangent vector ξ in the unit tangent bundle UM of (M, g), the geodesic γ with initial velocity vector $\dot{\gamma}(0) = \xi$ is

i) a periodic map with *least* period l (i.e. $\gamma(t + l) = \gamma(t)$ for every real number t and this does not hold for any l' in $]0, l[$);

ii) γ is *simple*, i.e. γ is injective on $[0, l[$.

SC^m: there exists a point m in M such that the above property is supposed to hold for every ξ in the unit sphere $U_m M$ at m (assumption on m only).

0.3. Then the ten thousand dollar question is: what can be said about an SC (or an SC^m) Riemannian manifold (M, g)? First it should be clear to the reader that in the SC^m case we cannot hope to get isometric or metric implications on M. For example,

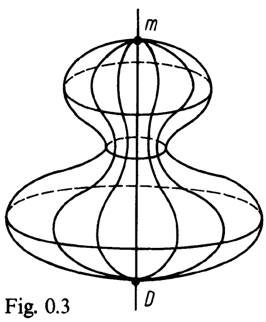


Fig. 0.3

any surface S of revolution in $\mathbb{R}^3$ about some axis D will satisfy the SC^m property if S meets D at some point m; but we can hope to be able to prove that a two-dimensional (M, g) with the SC^m property is necessarily homeomorphic or diffeomorphic to S^2 or $\mathbb{R}P^2$ (see 7.23).

On the contrary, if SC holds for (M, g) we can hope to prove that (M, g) is isometric to one of the CROSSes above.

History of the SC condition

0.4. It seems that the first historical attempt about the SC property is that of the pages 6—9 of Darboux [DX], who gave the explicit Condition 4.11 for the equation of a plane curve to generate by rotation around an axis a Riemannian surface (M, g) with the SC property. But Darboux did not establish the existence of such a globally defined metric on S^2 [of course non-isometric to the standard sphere (S^2, can)!].

0.5. In 4.27 the reader will find a pear constructed by Tannery. Every geodesic is closed and has least period l except the equator which has period l but least period $l/2$. Moreover, it has a singularity at one point. It is of historical interest to check the taste of mathematicians at the end of the century on this example. Tannery's pear is not a smooth manifold but it is algebraic! So this pear did not settle the question: does there exist a non-standard metric on S^2 with the SC property?

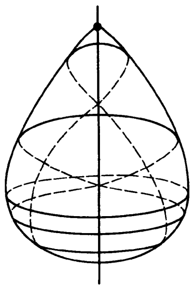


Fig. 0.5

0.6. The question was settled in 1903 by Zoll (see [ZL]) who gave an explicit (real analytic by the way) Riemannian manifold of revolution (S^2, g) with the SC property (see Chapter 4).

0.7. Having such a counter-example we are still far from the complete solution of the problem. A first question is: are there many non-isometric SC Riemannian manifolds (S^2, g)? But also secondly, are there such SC Riemannian structures on the other candidate manifolds $\mathbb{R}P^d$, $S^d (d \geqslant 3)$, $\mathbb{C}P^n (n \geqslant 2)$, $\mathbb{H}P^n (n \geqslant 2)$, $\mathbb{C}aP^2$?

Believe it or not (and in any case deduce from this fact that the problem is a very hard one despite its simple geometric formulation) nothing appeared in the literature before L. Green's theorem in 1961! Almost all of the main questions are open. The present state of the problem is as follows.

0.8. In 1913 Funk in [FK 1] studied the deformations of (S^2, can) by a one-parameter family $(S^2, g(t))$ with $g(0) = \text{can}$ and $g(t) = \varphi(t)\,\text{can}$, such that $g(t)$ is SC for every t.

What he essentially proved is the non-existence of such deformations under the additional condition that for every t the function $\varphi(t)$ on S^2 is *even* (i.e. invariant under the antipodal map) or, equivalently, that those $(S^2, g(t))$ are Riemannian coverings of a family of SC Riemannian metrics $\underline{g}(t)$ on the real projective space $\mathbb{R}P^2$. Otherwise stated: infinitesimally near the canonical one, there is no non-trivial SC Riemannian structure on $\mathbb{R}P^2$.

0.9. Moreover, Funk, in the same paper [FK 1], tried to construct a one-parameter family of SC metrics (S^2, $\varphi(t)$ can) with initial conditions $\varphi(0)=1$ and $\frac{d\varphi(t)}{dt}(0)=h$ (for any *odd* function h on S^2, i.e. satisfying $h\circ\sigma=-h$ for the antipodal map σ of S^2) as sum of a series. He failed to achieve convergence of his series. The existence of such deformations, for every odd initial derivative, has just been proved (1976) by V. Guillemin in [GU].

0.10. The existence of an SC structure itself on $\mathbb{R}P^2$ was studied by Blaschke in 1927 (see first edition of [BE] or [GN 2]), who conjectured that there is no non-trivial one. Truly speaking Blaschke was studying the problem lifted up to S^2 from the preceding one by the canonical covering $S^2 \to \mathbb{R}P^2$, see 0.30. Notice here again how long it took for mathematicians to really think about the abstract manifold $\mathbb{R}P^2$. This is confirmed by the fact that the elliptic geometry was founded long after the hyperbolic one (despite its greater simplicity). The non-existence of such non-trivial SC structures was proved by L. Green in 1961 (see [GN 2] or 5.59).

0.11. The existence of a nontrivial SC Riemannian metric g on S^d for $d \geqslant 3$ is not too hard to show (once one knows Zoll's example) and was settled by A. Weinstein (unpublished, see 4.E). At the moment of our writing, the problem of the existence of an SC Riemannian structure non-isometric to a CROSS on any of the manifolds $\mathbb{R}P^d$ $(d \geqslant 3)$, $\mathbb{C}P^n$ $(n \geqslant 2)$, $\mathbb{H}P^n$ $(n \geqslant 2)$, $\mathbb{C}a\,P^2$ is open. See however Appendix *D*.

0.12. However, in 1972 R. Michel in [ML 2] extended Funk's infinitesimal non-deformation result 0.8 to any dimension d for $\mathbb{R}P^d$. Otherwise stated: there is no hope of finding a non-trivial SC structure on $\mathbb{R}P^d$ for $d \geqslant 3$ by only an infinitesimal method.

0.13. The only general condition known in order that a Riemannian manifold (M, g) satisfy the SC property is the following result of A. Weinstein (see [WN 3] or 2.21): the total volume Vol(g) of a d-dimensional Riemannian manifold (M, g), which is a SC with period 2π, is an integral multiple of the volume $\beta(d)$ of (S^d, can).

History of the SC^m condition

0.14. As seen in 0.3 the realm of conclusions is now no longer Riemannian geometry but algebraic topology. What are the compact C^∞ manifolds M on which there exists a Riemannian metric g which makes (M, g) into an SC^m manifold? Are there others besides S^d, $\mathbb{R}P^d$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{C}aP^2$? The first and basic contribution on the subject is that of R. Bott in 1954 [BT], who proved that such an M should have the same integral cohomology ring $H^*(M; \mathbb{Z})$ as that of a CROSS, see also 7.23.

0.15. But we are still left with many problems since the above cohomology ring condition is far from a condition implying the same diffeomorphism, homeomorphism or homotopy type. For example any exotic sphere would do it. More precisely, L. Bérard Bergery found in 1975 an exotic sphere S^{10} with an SC^m Riemannian structure (see [B.B 1] or Appendix C). Other examples of manifolds with the same cohomology ring as that of a CROSS are the Eells-Kuiper exotic quaternionic projective planes [E-K], but up to the present day no SC^m Riemannian structure is known to exist on them.

0.16 Note. A completely different theme in Riemannian geometry is that of the existence of one, or many, closed geodesic(s) on a Riemannian manifold (M, g). For that subject we refer the reader to the very extensive and complete new book by W. Klingenberg [KG 2].

B. Organization and Contents

Chapter 1. Basic Facts about the Geodesic Flow

0.17. Despite our good intentions we could not write a self-contained book. However, we have put in Chapter 1 a good deal of the Riemannian geometry which is needed for the rest of the work. The unifying object in Chapter 1 is the unit tangent bundle of a Riemannian manifold, equipped with the geodesic flow. That emphasis on the unit tangent bundle is needed because it was the basic escalation of the second half of this century, which permitted Green's theorem, symplectic geometry (and we dare say Fourier integral operators!), etc. ... As a matter of fact L. Green was familiar with ergodic theory.

0.18. Chapter 1 is then quite developed and tries to help the reader become acquainted with the unit tangent bundle and the double tangent bundle. Foundations of Riemannian geometry are developed from the beginning, starting with the search for curves which minimize energy via the calculus of variations. One then gets the geodesic *spray* Z_g via the equation $i_{Z_g} d\alpha_g = -dE_g$ (cf. 1.47).

0.19. The problem here is to define the so-called Levi-Civita connection. We take this opportunity to prove that any spray gives rise to a unique symmetric connector and conversely. The method we adopt to construct the Levi-Civita connection is rather different from the one given in [BR 3], [G-K-M], or [K-N 1]. In particular the usual cyclic permutation trick which gives existence and uniqueness of the Levi-Civita covariant derivative does not play the same role here.

0.20. The end of the chapter is devoted to the Riemannian geometry of the unit tangent bundle, which is needed in Chapters 2 and 5.

Chapter 2. The Manifold of Geodesics

0.21. This chapter is devoted to the study of the manifold one can construct from the set CM of geodesics of a C manifold [i.e., a Riemannian manifold (M, g) with the same requirements SC as in 0.2 except that closed geodesics need not necessarily be simple curves]. The set CM has a natural structure of a C^∞ symplectic manifold, which moreover makes the map $q: UM \to CM$ into a fibration by circles. Here for ξ in UM $q(\xi)$ denotes the closed geodesic whose initial velocity vector at time 0 is precisely ξ. This is a particular case of a construction due to J.-M. Souriau [SU]. This construction is of great importance in mechanics and gives some new openings in order to study non-integrable Hamiltonian systems.

0.22. Chapter 2 systematically exploits the existence of CM and of the bundle $UM \to CM$; in particular, the various Riemannian structures that one can put—in a natural way— on CM and the Riemannian properties of these structures.

0.23. A basic fact is that the canonical 1-form of UM (see 1.58) turns out to be a connection form for the circle bundle $UM \to CM$, whose curvature form is simply the symplectic form of CM (see 2.11). It seems that this fact was exploited for the first time by Reeb in [RB 1] and it culminates in A. Weinstein's theorem 2.21 on the volume of a C manifold. Notice that the historical remark 0.10 can typically be applied here.

Chapter 3. Compact Symmetric Spaces of Rank one from a Geometric Point of View

0.24. We saw in 0.1 that compact symmetric spaces of rank one are our basic examples in the book. Thus they deserve an expository chapter. We think the reader will be glad to have most of the facts concerning them at hand, facts which come from various fields. This explains why almost any book on symmetric spaces is concerned with only one or two aspects. Fields coming into the picture are: homogeneous spaces of Lie groups, Riemannian geometry via the covariant derivative of the curvature tensor and also via the geodesic symmetries around points (hence the name symmetric spaces!), projective geometry and analysis to study the spectrum of the Laplacian.

0.25. We have tried to give a lucid exposition centered on projective spaces, carefully expliciting in the homogeneous space framework the isotropy action (which to us did not seem well described in the literature). Also we have carefully written down the two-fold inheritance: two kinds of inclusions, one from $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{C}\mathrm{a}$ and the other one $\mathbb{K}P^n \subset \mathbb{K}P^{n'}$ $(n \leqslant n')$. Finally we include an explicit classification of all totally geodesic submanifolds of our CROSS'es.

0.26. All of this program is not too hard to carry out for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or for the spheres. However we encounter a prime difficulty when defining $\mathbb{C}\mathrm{a}\,P^2$. The reason for this is the nonexistence of a Hopf fibration $S^{23} \to \mathbb{C}\mathrm{a}\,P^2$ with fiber S^7 which would allow us to define $\mathbb{C}\mathrm{a}\,P^2$ as a suitable quotient of S^{23} (as we did for the other $\mathbb{K}P^n$). The basic reason for this non-existence is the fact that the system of Cayley numbers $\mathbb{C}\mathrm{a}$ is not associative. This difficult is illustrated by the following two dependent observations: firstly there is no printed text in which $\mathbb{C}\mathrm{a}P^2$ is constructed and studied in detail and secondly the number of young (or aged) geometers asking us for references on this wonderful but somewhat frustrating $\mathbb{C}\mathrm{a}P^2$ is fairly large.

Chapter 4. Some Examples of C- and P-Manifolds: Zoll's and Tannery's Surfaces

0.27. If Chapter 3 considers the *classical* examples of *SC* manifolds, Chapter 4 studies the 2-dimensional *exotic* examples, namely those of Zoll and of V. Guillemin. Zoll's original construction (see [ZL] or [BR 3]) was almost a miracle: the meridian curve was built up in two pieces Γ', Γ'' whose union is real analytic (believe it or not!).

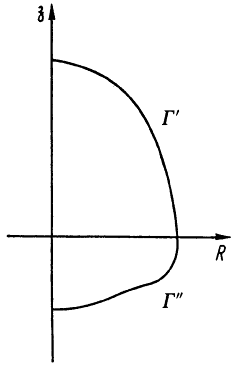


Fig. 0.27

0.28. We give an abstract expression for the *SC* Riemannian metrics g on S^2 which admit an S^1 action by isometries. This expression is very simple and involves an arbitrary odd real-valued function on $[0, \pi]$. It extends Zoll's example to a wider class and also avoids the above tedious construction.

We also give various properties of these Zoll metrics and in particular an explicit $\mathbb{R}^3$-embedding for some of them.

0.29. The chapter ends with Guillemin's result quoted in 0.9, see 4.H. The proof is only sketched, the reader being referred to [GU] because it is fairly technical. We only note that V. Guillemin's result proves, among other facts, that there are many *SC* metrics on S^2 with no isometries but the identity. It is in order to quote here Gambier (1925, see [GR] or 4.C) and Funk (1923, see [FK 2] or 4.B) who studied this problem with some success. We finish with a proof following A. Weinstein [WN 4] that the geodesic flows of two *SC* metrics on S^2 are conjugate by a symplectic diffeomorphism.

Chapter 5. Blaschke Manifolds and Blaschke's Conjecture

0.30. There is a striking difference between the behaviour of geodesics of (S^2, can) and those of a Zoll surface. In (S^2, can) all geodesics starting from a given point m pass at time π the same point m', namely the antipod of m. But in a Zoll surface of revolution the geodesics, starting from a point m which is not on the axis of revolution, have in general a non-trivial envelope Γ:

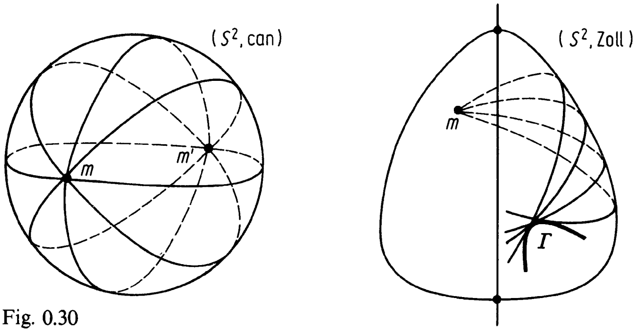


Fig. 0.30

The problem mentioned in 0.10 and studied by Blaschke was the following: suppose that (S^2, g) is a Riemannian manifold such that every geodesic from any point m is at time π at a constant point m' different from m. Is then (S^2, g) necessarily isometric to $(S^2$, can)? We mentioned in 0.10 that this problem is equivalent to studying SC structures on $\mathbb{R}P^2$.

0.31. More generally it is natural to study Riemannian manifolds (M, g) whose geodesic behaviour is that of a CROSS in the following sense (compare with 3.35): for every pair of points (m, n) in M, such that n belongs to the cut-locus Cut (m) of m, the shortest geodesics (called segments, see 5.12) from m to n have velocity vectors in U_nM which form a whole great sphere of U_nM (see 5.36). The case of only two antipodal points in U_nM is that of ($\mathbb{R}P^2$, can); that of all possible directions is the generalization of Blaschke's case. For ($\mathbb{C}P^n$, can) one gets a circle of the Hopf fibration of U_nM:

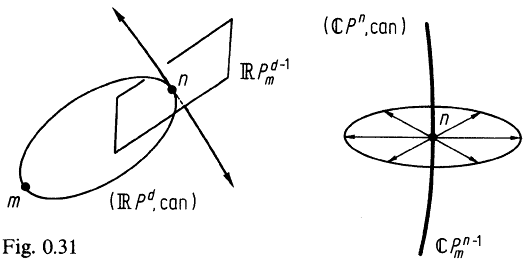


Fig. 0.31

0.32. Chapter 5 is devoted to the study of Riemannian manifolds (M, g) with such a geodesic structure. They are called *Blaschke manifolds*, and *Blaschke manifolds at m* if the above condition is required to hold only for every point n in the cut-locus of some point m in M. They will be characterized by two other equivalent conditions: the first one is that the distance from m to any point n in its cut-locus is constant. The second one is the fact that M can be written as $D \bigcup_a E$, where D is a closed d-dimensional ball, E is a closed disc bundle over a nice compact manifold, and a is an attaching diffeomorphism between the boundary of D and that of E (a boundary which has then to be homeomorphic to S^{d-1}).

0.33. Blaschke's conjecture is that any Blaschke manifold is isometric to a CROSS. L. Green's theorem mentioned in 0.10 proves the conjecture when the dimension is equal to 2. For dimensions $d \geqslant 3$ the conjecture is completely open. However, we shall give Michel's theorem, which is the extension to any dimension d of Funk's result 0.8. See also Appendix D.

Chapter 6. Harmonic Manifolds

0.34. Harmonic manifolds "slept" in this book by the side door. Their definition has nothing to do with closed geodesics. For a Riemannian manifold (M, g), a point m in M and a small enough real number r the geodesic sphere $S(m, r)$ is a nice submanifold. To compute its volume we can work in polar coordinates centered at m. The volume of $S(m, r)$ will be given by the formula $\mathrm{Vol}\,(S(m, r)) = \int_{U_m M} \delta(\xi, r) d\sigma$ where ξ denotes a unit vector in $U_m M$ and σ the canonical measure on $U_m M$ (see 1.120). Then we say that (M, g) is locally harmonic if the density $\delta(\xi, r)$ depends only on r and not on ξ or on m. One also defines a notion of global harmonicity.

0.35. The problem of finding all locally or globally harmonic manifolds is completely open, except for dimensions less than 5. But a result of Allamigeon implies that compact and simply connected globally harmonic manifolds are Blaschke manifolds (with of course additional properties). That is why we have decided to include harmonic manifolds in this book.

0.36. The beginning of the chapter gives the local and global definitions above, together with a third one: that of strongly harmonic manifolds, which requires compactness and involves the heat equation or, equivalently the spectrum of the Laplacian and its eigenfunctions. Equivalence of these notions is studied in detail.

0.37. Harmonic manifolds necessarily satisfy curvature conditions. The first condition yields the fact that such a manifold should be an Einstein manifold, the second is quadratic in the curvature tensor R, the third is cubic in R and quadratic in its covariant derivative DR. The other conditions are so complicated that nobody has ever worked them out. From the curvature conditions above we prove the Lichnerowicz-Walker theorem: harmonic manifolds of dimension less than or equal to 4 are locally isometric to a ROSS or are flat.

0.38. Allamigeon's theorem is proved essentially in Chapter 5. At the end of Chapter 6 we study strongly harmonic manifolds. They can be embedded as minimal submanifolds in Euclidean spheres and moreover these embedded submanifolds have all their geodesics congruent as curves in the Euclidean space. This last condition is a good support for conjecturing that simply connected globally harmonic manifolds are isometric to CROSSes.

Chapter 7. On the Topology of SC- and P-Manifolds

0.39. Roughly speaking this chapter concerns SC^m manifolds (for a definition see 0.14 and 0.15) and gives the proof of R. Bott's theorem. But in fact, when thinking of weakening the very strong SC^m assumption and also studying links between SC and SC^m conditions, one encounters disguised difficulties. An example might help the reader: consider (S^3, can) and a subgroup G of the special orthogonal group $SO(4)$ which acts on S^3 without fixed point. Then we can construct a quotient Riemannian manifold $(M, g) = (S^3/G, \text{can}/G)$, called a *lens space*. On (M, g) every geodesic is periodic with period 2π as is the case on (S^3, can). But 2π is not the least period of these geodesics, some have a least period equal to $2\pi/k$ for some integer $k > 1$.

0.40. *Worse*: if (M, g) is a Riemannian manifold with the property that every geodesic is periodic with a certain period, it is not obvious that there is a *common* period for all these geodesics. This is in fact a very nice result due to Wadsley ([WY 2]) (it is included in Appendix A, by D. Epstein, for the convenience of the reader). Notice that this result, considered as a result about one-dimensional foliations (this is precisely the case for the geodesic flow on UM) is only valid for geodesic foliations; namely our assumption transferred to the unit tangent bundle UM is precisely that UM admits a geodesic foliation with compact leaves (the integral curves of the geodesic flow). The desired conclusion about a common period is equivalent to the boundedness of the length of the leaves. But for a non-geodesic foliation of dimension one on a compact manifold, there is no such result since Thurston has constructed an example of a compact 5-dimensional manifold with a one-dimensional foliation whose leaves are compact but have unbounded lengths. This example will be sketched in Appendix A.

0.41. We hope we have convinced the reader of the existence of snags hidden in the subject. Then he will understand the necessity at the beginning of Chapter 7 of formalizing how a manifold can have all its geodesics closed. After these definitions, we give some non-trivial implications between them and some open problems on the subject. A weakened form of R. Bott's theorem is proved for a class of manifolds all of whose geodesics are closed (namely P-manifolds).

0.42. The chapter ends with the complete classification of homogeneous manifolds all of whose geodesics are closed: up to isometry these are the CROSSes.

Chapter 8. The Spectrum of P-Manifolds

0.43. This chapter is mainly dedicated to a proof of the surprising Duistermaat-Guillemin result ([D-G]) to the effect that P_l-manifolds (i.e. Riemannian manifolds such that every geodesic admits l as *a* period) can be characterized as those compact Riemannian manifolds for which the square root of the spectrum of the Laplace operator Δ is asymptotically an arithmetic progression (for example Zoll's surfaces will necessarily have such a spectrum). The initial value and the ratio of the arithmetic sequence yield exactly the least common period l and modulo 4 the Morse index of our closed geodesics. The chapter ends with a property of the first eigenvalue λ_1 of Δ for an SC structure on S^d.

Appendices

0.44. We have added to the chapters of the book five appendices on related topics. Appendix A, written by David Epstein, presents a proof of Wadsley's result 0.40 and contains a brief description of Thurston's counter-example 0.40.

0.45. Appendix B by Jean-Pierre Bourguignon was motivated by the following question which arises naturally when studying SC manifolds: Consider a Riemannian metric on S^2 with the Blaschke property mentioned in 0.30 (i.e. all geodesics from m meet again at time π at a different point m', and not before). If we take the transverse derivative of this condition along a fixed geodesic γ, the general philosophy of Jacobi fields will yield the following consequence: for every t, the Jacobi field Y_t along γ defined by $Y_t(t)=0$ and $Y_t'(0)=1$ will satisfy $Y_t(t+\pi)=0$ and $Y_t(s)\neq 0$ for every s in $]t, t+\pi[$. One way to prove L. Green's theorem (cf. 0.10) would be to prove that the corresponding Jacobi equation $Y''+\sigma Y=0$, where σ is the curvature, is necessarily $Y''+Y=0$, namely that (S^2, g) has constant curvature hence is isometric to (S^2, can). But only quite recently did some analysts give the explicit general form of functions σ such that the equation $Y''+\sigma Y=0$ has only antiperiodic solutions. Appendix B gives this explicit form (which is due to [NN]) and also some considerations about the case of a system (with the higher dimensional Blaschke conjecture in mind).

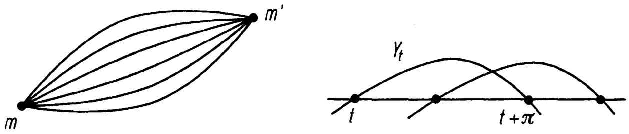


Fig. 0.45

0.46. Appendix C by Lionel Bérard Bergery gives the construction (cf. 0.15 and 0.32) of a Blaschke structure at the center m of a disk D on a compact manifold $M=D\bigcup_a E$. The existence of a Riemannian metric g on $M=D\bigcup_a E$ such that all geodesics from m are not only loops but are *closed* geodesics is then studied. Finally an example of an exotic sphere with such a Riemannian structure is given.

0.47. Appendix D by Marcel Berger contains the proof of Blaschke's conjecture for spheres and real projective spaces. The proof rests eventually on an inequality proved by Jerry L. Kazdan in Appendix E.

C. What is New in this Book?

For the convenience of the reader and in order to be fair we give here a list of what is really new in the book.

It should be clear that these new things are at various levels of difficulty.

Chapter 1. An exposition of the foundations of both Riemannian and symplectic geometry starting from the geodesic flow, an intrinsic version of the relationship between connectors and sprays.

Chapter 2. A systematic treatment of the manifold of geodesics of a C manifold, of its Riemannian geometry and in particular of its geodesics and its curvature.

Chapter 3. A lucid exposition (one must admit not always with detailed proofs) of the Cayley projective plane, with a good set of references; and a presentation of the symmetric spaces of rank one with a detailed exposition of the isotropy group and their two-fold hereditary character (change of dimension and change of field).

Chapter 4. The complete determination of all Zoll metrics on S^2 which are invariant under an S^1 action, and their realization from their abstract expression as equivariant imbeddings in $\mathbb{R}^3$; explicit computation of some of these examples, their cut-loci and conjugate loci.

Chapter 5. A new proof of Michel's theorem, whose original proof was quite difficult and somewhat mysterious.

Chapter 6. The introduction of various definitions of harmonicity on a Riemannian manifold and the relations between these definitions in the differentiable case. In particular the introduction of the notion of strong harmonicity and the fact that a strongly harmonic manifold admits isometric imbeddings in Euclidean spheres S, which yields minimal submanifolds in S all of whose geodesics are congruent curves of the Euclidean space.

Chapter 7. The introduction of various definitions for manifolds "all of whose geodesics are closed" and various implications among these definitions. A proof of a Bott's type theorem for P-manifolds. The complete classification of homogeneous manifolds all of whose geodesics are closed.

Chapter 8. The proof which we give for the Theorem 8.9 (Duistermaat-Guillemin) is slightly different from the original one. The result in Proposition 8.47 is new, at least to our knowledge.

Appendix A. An improved proof of A. Wadsley's theorem (see [WY 2]).

Appendix B. Contains a slightly different method of proving L. Green's theorem (see B.25).

Appendix C. A new proof of A. Weinstein's converse to the Allamigeon-Warner theorem, using Riemannian submersion techniques. An example of an exotic sphere with a Riemannian metric such that for some point every geodesic through that point is periodic.

Appendix D. An extension of L. Green's Theorem to every dimension (see 0.10, 0.33).

Appendix E. An integral inequality used in Appendix D.

D. What are the Main Problems Today?

In various chapters the reader will find many problems, in particular in 5.74—79, 6.16, 6.88, 6.105, 7.66. However, for his convenience we extract here what are, we think, the main ones.

Pb. 1: classify C-structures on S^d ($d=2$ and $d>2$) (do they form a nice subset of Riemannian metrics on S^d?).

Pb. 2: do there exist non-canonical C-structures on $\mathbb{C}P^n$ ($n \geqslant 2$), $\mathbb{H}P^n$ ($n \geqslant 2$), $\mathbb{C}a\, P^2$? If yes, classify them.

Pb. 3: settle Blaschke's conjecture.

Pb. 4: do there exist C-structures with a Weinstein integer different from the standard one?

Pb. 5: what exactly are the manifolds admitting a C-structure or a P-structure?

Pb. 6: what is the right metric on C^gM (the manifold of geodesics)?

Pb. 7: does "P" imply "C" for simply connected manifolds?

Pb. 8: extend R. Michel's result 5.90 to more general deformations on $\mathbb{C}P^n$ and to other CROSSes.

Pb. 9: find all harmonic manifolds (infinitesimally, locally, globally, strongly).

Pb. 10: classify the geodesic flow associated to C_l-metrics from a symplectic point of view (see 8.43 for the case of S^2).

Pb. 11: generalize 8.47 to higher dimension.

Note that Pb. 9 is a subproblem of what is a challenging problem in Riemannian geometry at the moment: find all compact Einstein manifolds.