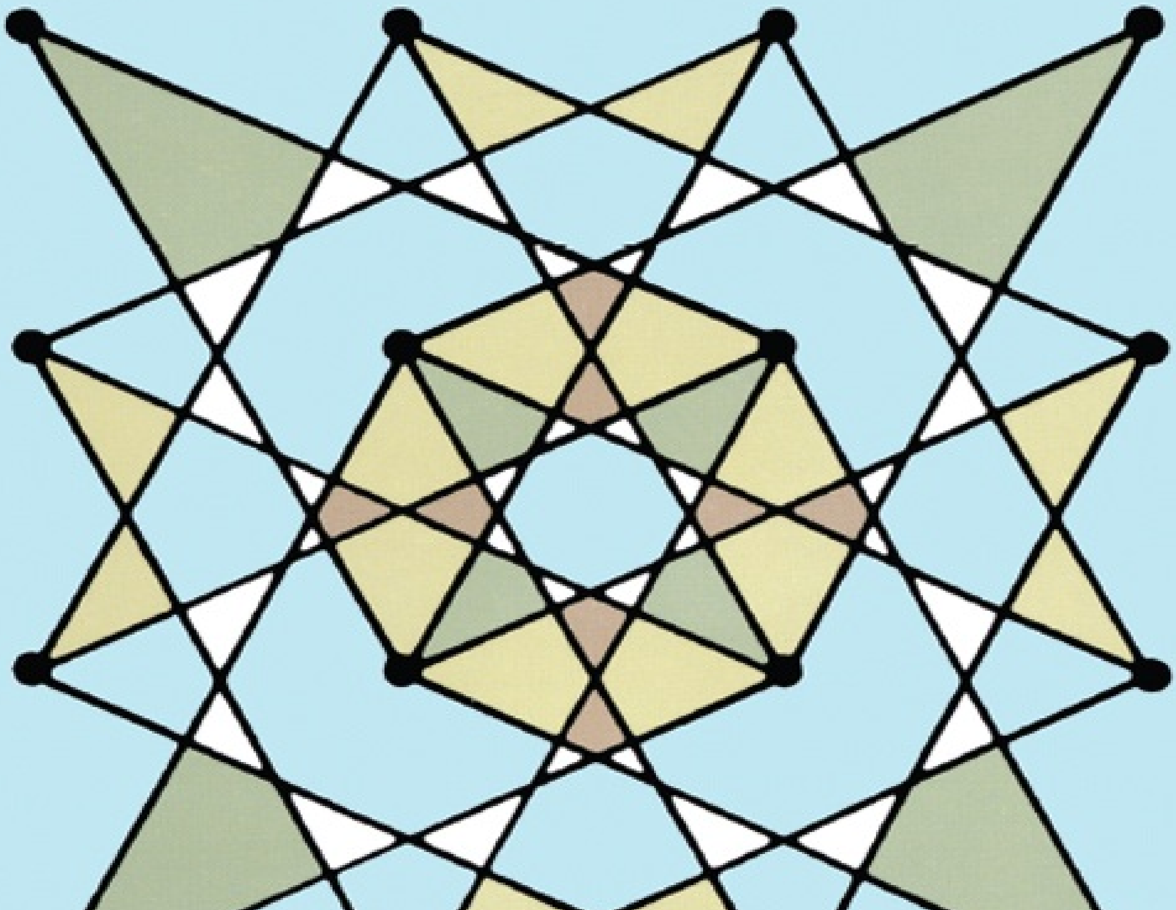
INTRODUCTION TO GRAPH THEORY



Richard J. Trudeau

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To Dick Barbieri and Chet Raymo

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PREFACE

This book is about pure mathematics in general, and the theory of graphs in particular. ("Graphs" are networks of dots and lines;* they have nothing to do with "graphs of equations.") I have interwoven the two topics, the idea being that the graph theory will illustrate what I have to say about the nature and spirit of pure mathematics, and at the same time the running commentary about pure mathematics will clarify what we do in graph theory.

I have three types of reader in mind.

First, and closest to my heart, the mathematically traumatized. If you are such a person, if you had or are having a rough time with mathematics in school, if you feel mathematically stupid but wish you didn't, if you feel there must be *something* to mathematics if only you knew what it was, then there's a good chance you'll find this book helpful. It presents mathematics under a different aspect. For one thing, it deals with *pure* mathematics, whereas school mathematics (geometry excepted) is mostly *applied* mathematics. For another, it is a more qualitative than quantitative study, so there are few calculations.

Second, the mathematical hobbyist. I think graph theory makes for marvelous recreational mathematics; it is intuitively accessible and rich in unsolved problems.

Third, the serious student of mathematics. Graph theory is the oldest and most geometric branch of topology, making it a natural supplement to either a geometry or topology course. And due to its wide applicability, it is currently quite fashionable.

The book uses some algebra. If you've had a year or so of high school algebra that should be enough. Remembering specifics is not so important as having a general familiarity with equations and inequalities. Also, the discussion in Chapter 1 presupposes some experience with plane geometry. Again no specific knowledge is required, just a feeling for how the game is played.

Chapter 7 is intended for the more mathematically sophisticated reader. It generalizes Chapters 3–6. It is more conceptually difficult and concisely written than the other chapters. It is not, however, a prerequisite for Chapter 8.

The exercises range from trivial to challenging. They are not arranged in order of difficulty, nor have I given any other clue to their difficulty, on the theory that it is worthwhile to examine them all.

The suggested readings are nontechnical. Those that have been starred are available in paperback.

There are a number of more advanced books on graph theory, but I especially recommend *Graph Theory* by Frank Harary (Addison-Wesley, 1969). It contains a wealth of material. Also, graph theory's terminology is still in flux and I have modeled mine more or less after Harary's.

Richard J. Trudeau
July 1975

* This book was originally published under the title *Dots and Lines*.

1. PURE MATHEMATICS

Introduction

This book is an attempt to explain pure mathematics. In this chapter we'll talk about it. In Chapters 2–8 we'll *do* it.

Most pre-college mathematics courses are oriented toward solving "practical" problems, problems like these:

> A train leaves Philadelphia for New York at 3:00 PM and travels at 60 mph. Another train leaves New York for Philadelphia at 3:30 PM and travels at 75 mph. If the distance between the cities is 90 miles, when and at what point will the trains pass?

> If a 12-foot ladder leaning against a house makes a 75° angle with the ground, how far is the foot of the ladder from the house and how far is the top of the ladder from the ground?

Mathematics that is developed with an eye to practical applications is called "applied mathematics". With the possible exception of Euclidean geometry, pre-college mathematics is usually applied mathematics.

There is another kind of mathematics, called "pure mathematics", which is a charming little pastime from which some people derive tremendous enjoyment. It is also the basis for applied mathematics, the "mathematics" part of applied mathematics. Pure mathematics is *real* mathematics.

To understand what mathematics is, you need to understand what pure mathematics is. Unfortunately, most people have either seen no pure mathematics at all, or so little that they have no real feeling for it. Consequently most people don't really understand mathematics; I think this is why so many people are afraid of mathematics and quick to proclaim themselves mathematically stupid.

Of course, since pure mathematics is the foundation of applied mathematics, you can see the pure mathematics beneath the applications if you look hard enough. But what people see, and remember, is a matter of emphasis. People are told about bridges and missiles and computers. Usually they don't hear about the fascinating intellectual game that lies beneath it all.

Earlier I implied that Euclidean geometry—high school geometry— might be an example of pure mathematics. Whether it is or not again depends on emphasis.

Euclidean geometry as pure mathematics

What we call "Euclidean geometry" was developed in Greece between 600 and 300 B.C., and codified at the end of that period by Euclid in *The Elements. The Elements* is the archetype of pure mathematics, and a paradigm that mathematicians have emulated ever since its appearance. It begins abruptly with a list of definitions, followed by a list of basic assumptions or "axioms" (Euclid states ten axioms, but there are others he didn't write down). Thereafter the work consists of a single deductive chain of 465 theorems, including not only much of what was known at that time of geometry, but algebra and number theory as well. Though that's quite a lot for one book, people who read *The Elements* for the first time often get a feeling that things are missing: it has no preface or introduction, no statement of objectives, and it offers no motivation or commentary. Most strikingly, there is no mention of the scientific and technological uses to which many of the theorems can be put, nor any warning that large sections of the work have no practical use at all. Euclid was certainly aware of applications, but for him they were not an issue. To Euclid a theorem was significant, or not, in and of itself; it did not become more significant if applications were discovered, or less so if none were discovered. He saw applications as external factors having no bearing on a theorem's inherent quality. The theorems are included *for their own sake,* because they are interesting in themselves. This attitude of self-sufficiency is the hallmark of pure mathematics.

The Elements is the most successful textbook ever written. It has gone through more than a thousand editions and is still used in some parts of the world, though in this country it was retired around the middle of the nineteenth century. It is amazing that it was used as a school text at all, let alone for 2200 years, as it was written for adults and isn't all that easy to learn from.

Of the modern texts that have replaced *The Elements*, many are faithful to its spirit and present geometry as pure mathematics, *ars gratia artis.* There are others, however, that have sandwiched around Euclid's work long discussions of the "relevance" and "practicality" of geometry, complete with pictures of office buildings and space capsules and the suggestion that geometry is primarily a branch of engineering. Of course, applied geometry is a perfectly valid science, but I can't help objecting to such books on the ground that they rob school children of their only encounter with pure mathematics.

Despite this problem, in the next section I shall use Euclidean geometry as an example of pure mathematics, as it is the branch of pure mathematics with which you are most likely to be familiar.

Games

Basically pure mathematics is a box of games. At last count it contained more than eighty of them. One of them is called "Euclidean geometry". In this section I will compare Euclidean geometry to chess, but you won't have to be a chessplayer to follow the discussion.

Games have four components: objects to play with, an opening arrangement, rules, and a goal. In chess the objects are a chessboard and chessmen. The opening arrangement is the arrangement of the pieces on the board at the start of the game. The rules of chess tell how the pieces move; that is, they specify how new arrangements can be created from the opening arrangement. The goal is called “checkmate” and can be described as an arrangement having certain desirable properties, a “nice” arrangement.

In Euclidean geometry the *objects* are a plane, some points, and some lines. The plane corresponds to a chessboard, the points and lines to chessmen. The *opening arrangement* is the list of axioms, which are accepted without proof. The analogy with the opening arrangement of chessmen may not be apparent, but it is quite strong. First, the opening arrangement of chessmen is *given;* to play chess you must start with that arrangement and no other. In the same way the axioms of Euclidean geometry are given. Second, the opening arrangement of chessmen specifies how the objects with which the game is played are related at the outset. This is exactly what the axioms do for the game of Euclidean geometry; they tell us, for example, that points and lines lie in the plane, that through two points there passes one and only one line, etc. The *rules* of Euclidean geometry are the rules of formal logic, which is nothing but an etherealized version of the “common sense” we absorb from the culture as we grow up. Its rules tell us how statements can be combined to produce other statements. They tell us, for example, that the statements “All men are mortal” and “Plato is a man” yield the statement “Plato is mortal.” (The example is Aristotle’s. He wrote the first book on logic by recording patterns of inference he saw people using every day.) In particular the rules of logic tell us how to create, from the opening arrangement (the list of axioms), new arrangements (called “theorems”). And the *goal* of Euclidean geometry is to produce as many “nice” arrangements as possible, that is to prove profound and surprising theorems. Checkmate terminates a chessmatch, but Euclidean geometry is open-ended.

Games have one more feature in common with pure mathematics. It is subtle but important. It is that the objects with which a game is played have no meaning outside the context of the game.

Chessmen, for example, are significant only in reference to chess. They have no necessary correspondence with anything external to the game. Of course, we could *interpret* the pieces as regiments at the First Battle of the Marne, and the board as French countryside. Or an interpretation could be brought about by a wager, say each piece represents $5.00 and losing it means paying that amount. But no such correspondence between the game and things outside the game is necessary.

You may balk at this, since, for historical reasons, chessmen have names and shapes that imply an essential correspondence with the external world. The key word is “essential”; there is indeed a correspondence, but it is inessential. After all, chessmen require names of some sort, and must be shaped differently to avoid confusion, so why not call them “kings”, “queens”, “bishops”, “knights”, etc., shape them accordingly, and trade on the image of excitement and competition thereby created? Doing so is harmless and makes the game more popular. But this particular interpretation of the game, like all others, has nothing to do with chess *per se.*

Here’s an example. Suppose we substitute silver dollars for kings, half-dollars for queens, quarters for bishops, dimes for knights, nickels for rooks, pennies for pawns, and an eight-by-eight array of chartreuse and violet circles for the standard board, but otherwise follow the rules of chess. Such a game would look strange and even sound strange—“pawn to king four” would now be “penny to silver dollar four”—but surely if we were to play this apparently unfamiliar game, there would be no doubt that we are playing the familiar game of chess. Indeed, chessmasters sometimes play without a board or pieces of any kind; they merely announce the moves and keep track in their heads. Two such people are still playing chess, for after all they say they are, and they certainly should know.

It appears then that the essence of chess is its abstract structure. Names and shapes of pieces, colors of squares, whether the “squares” are in fact square, even the physical existence of board and pieces, are all irrelevant. What is relevant is the number and geometric arrangement of the “squares”, the number of types of piece and the number of pieces of each type, the quantitative-geometric power of each piece, etc. Everything else is a visual aid or a fairy tale.

So it is with pure mathematics. Euclid’s words “plane”, “point”, and “line” suggest that geometry deals with flat surfaces, tiny dots, and stretched strings, but this implied interpretation of geometry is only that. It is analogous to the interpretation of chess as a battle. Geometry is no more a study of flat surfaces and dots than chess is a military exercise. As in any game, the objects geometers play with, and consequently their arrangements—the axioms and theorems—have no necessary correspondence with things external to the game.

In support of this let me point out that geometers never define the words “plane”, “point”, or “line”. (Euclid offered an intuitive explanation but did not actually define them; moderns leave the words undefined.) So *no one knows* what planes, points, or lines are, except to say that they are objects which are related to one another in accordance with the axioms. The three words are merely convenient names for the three types of object geometers play with. Any other names would do as well. Were we to attack Euclid’s *Elements* with an eraser and remove every occurrence of the words “plane”, “point”, and “line”, replacing them respectively with the symbols “#”, “$”, and “?”, the result would still be *The Elements* and the game would still be geometry. To a casual observer the vandalized *Elements* wouldn’t look like geometry; what had been “two points determine one and only one line” would now be “two $’s determine one and only one ?”. But then two people hunched over a board of chartreuse and violet circles, littered with coins, doesn’t look like a chessmatch. The game would still be geometry because it would be structurally identical to geometry. And were we to further maim *The Elements* by erasing all the diagrams, it still wouldn’t make a difference. Geometric diagrams are to geometers what board and pieces are to chessmasters: visual aids, helpful but not indispensable.

Why study pure mathematics?

There emerges from the foregoing an image of pure mathematics as a meaningless intellectual pastime. Yet carved over the door to Plato’s Academy was the admonition, “Let no one ignorant of geometry enter here!” And pure mathematics has been held in the highest regard ever since. It would seem to have no more to recommend its inclusion in school curricula than, for instance, chess, yet it is universally favored by academics over other games. I shall give three reasons for this.

Pure mathematics is applicable. Because pure mathematics has no inherent correspondence with the outside world, we are free to make it correspond, to interpret it, in any way we choose. And it so happens—this is the interesting part—that most branches of pure mathematics can be interpreted in such a way that the axioms and theorems become approximately true statements about the external world. In fact, some branches have several such interpretations.

Pure mathematics that has been made to correspond in this way to the world outside is called "applied mathematics". Pure mathematics is Euclid saying "three $'s not on the same ? determine a unique #." Applied mathematics is a surveyor reading Euclid, interpreting and "#" in a way that seems in accord with the axioms, and concluding that a tripod would be the most stable support for his telescope.

On one level, the applicability of pure mathematics is no surprise. Just as chess (as we know it) has been modeled on certain aspects of medieval warfare, even though strictly speaking the game has nothing to do with warfare, so too most branches of pure mathematics have started as models of physical situations. A branch of pure mathematics utterly lacking in significant interpretations would be boring to the community of pure mathematicians and would soon die out from lack of interest. Though they are unconcerned with applications as such, pure mathematicians are like most people in that they find it hard to be enthusiastic about something unless, under some aspect at least, it has the spontaneous appearance of truth.

But on a deeper level, the applicability of pure mathematics is quite mysterious. It's true that pure mathematics often originates with an abstraction from the physical world, as geometry begins with idealized dots and strings and tabletops, but the tie is only historical. Once the abstractions have been made the mathematical game comes into independent existence and evolves under its own laws. It has no necessary correspondence with the original physical situation. The mathematician does not deal with physical objects themselves, but with idealizations that exist independently and differ from their physical counterparts in a great many respects. And entirely within his own mind, the mathematician subjects these abstractions to a reflective, self-analytic process, a process in which he is trying to learn about himself, to learn what in a sense he already knows. This process is strictly internal to a human mind—a Western mind at that—and so is presumably different from whatever the process by which the physical situation evolves; yet when the mathematician compares his results to outside events, he often finds that nature has evolved to a state remarkably like his mathematical model. That the universe is so constructed has seemed uncanny to many famous mathematicians and scientists, moving them to comment in a mystical fashion that seems totally out of character:

> "Number rules the universe."
>
> —Pythagoras

> "Mathematics is the only true metaphysics."
>
> —Lord Kelvin

> "How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?"
>
> —Einstein

> “The Great Architect of the Universe now begins to appear as a pure mathematician. ”
>
> —Sir James Jeans

(Quotations from E. T. Bell’s *Men of Mathematics.)*

Applicability is the chief difference between the games known collectively as “pure mathematics” and other games. There’s some kind of chemistry involving nature, and people, and pure mathematics, that enables applied mathematics to predict the future, whereas mankind has yet to make a success of applied chess.

Pure mathematics is a culture clue.

> “. . . common sense is, as a matter of fact, nothing more than layers of preconceived notions stored in our memories and emotions for the most part before age eighteen.”
>
> —Albert Einstein

Our common sense, or world view, is not “common” to all people. It is shaped by the culture we inhabit. It is like a pair of glasses few of us ever manage to take off, so of course we see confirmation everywhere we look.

Much of Western intellectual tradition has been inherited from the Greeks. Our science and philosophy in particular are shot through with beliefs and opinions and forms of speech that were once explicit doctrines of Plato, Aristotle, and the like, but have come to be embedded anonymously in the fabric of our thought. Of this embedded material perhaps the most fundamental is logic, the standard by which we judge reasoning to be “correct”, a standard first written down by Aristotle in *The Organon* (about 350 B.C.).

Is logic itself “correct”? Some Eastern philosophers would call it “ignorance”. I use logic all the time in mathematics, and it seems to yield “correct” results, but in mathematics “correct” by and large means “logical”, so I’m back where I started. I can’t defend logic because I can’t remove my glasses.

“Correct” or not, logic is basic to Western rationality and to the whole scientific enterprise. And not surprisingly, since logic is the study of deduction and pure mathematics is the only completely deductive study, logic is inextricably intertwined with pure mathematics. I think this is the chief reason for the prominence of mathematics in our schools. Logic is a fundamental component of the culture, so the culture quite naturally sets a premium on teaching the next generation to think in logical categories.

Incidentally, there’s a lot of debate on which came first, logic or mathematics. In one sense logic is prior to mathematics, as mathematics uses the laws of logic. But Aristotle abstracted the laws of logic at least in part from the pure mathematics he studied at Plato’s Academy, so in another sense mathematics is more basic. G. Spencer Brown argues this position in *Laws of Form,* p. 102:

> A theorem is no more proved by logic and computation than a sonnet is written by grammar and rhetoric, or than a sonata is composed by harmony and counterpoint, or a picture painted by balance and perspective. Logic and computation, grammar and rhetoric, harmony and counterpoint, balance and perspective, can be seen in the work *after* it is created, but these forms are, in the final analysis, parasitic on, they have no existence apart from, the creativity of the work itself. Thus the relation of logic to mathematics is seen to be that of an applied science to its pure ground, and all applied science is seen as drawing sustenance from a

process of creation with which it can combine to give structure, but which it cannot appropriate.

Pure mathematics is fun. At this moment there are thousands of people around the world doing pure mathematics. A few might be doing so because they foresee a possible application. A few might be philosophers taking Bertrand Russell's advice that "to create a healthy philosophy you should renounce metaphysics but be a good mathematician." There might even be a few ascetics who are doing it to sharpen their minds. But the vast majority are doing it simply because it's fun.

Pure mathematics is a first-rate intellectual adventure, ". . . an independent world/Created out of pure intelligence" (Wordsworth) that is neither science nor art but somehow partakes of both.

Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere—Archimedes did it in a bathtub. It is dramatic, challenging, endless, and full of surprises.

Pure mathematics is a pleasant way to pass the time until the end. And to me that makes it very serious, very important indeed.

What's coming

Talking about pure mathematics isn't enough. To really understand what it is, you have to do it. That's why this book has seven more chapters.

In the pages ahead we shall develop the rudiments of one rather modest game from the pure mathematics game-box. It has the unfortunate name "graph theory". The name is unfortunate because it is misleading. The objects with which the game is played are called "graphs", which is also the name given to the pictures of equations drawn in high school algebra courses. "Graph" in our sense of the word is not a picture of an equation, but rather a network of dots and lines. "Network theory" would be a better name for our game. Nevertheless the official name of the game is "graph theory", and mathematicians just have to remember that the "graph" of "graph theory" is not an algebraic graph. You'll appreciate the difference after reading Chapter 2.

Pure mathematics is a *big* game-box. I selected graph theory because it has several features that seemed desirable for this kind of book:

Graph theory is new; the bulk of it has been developed since 1890.

Graphs are simple, intuitively accessible things. In a book of this size we can develop enough graph theory to prove several spectacular theorems, and to half-prove several others.

Graph theory's frontier is easily reached, and you will find yourself, probably for the first time, thinking about famous unsolved problems.

If, like so many people, you dislike mathematics, it may be that you merely dislike applied mathematics, in which case you'll enjoy working your way through the chapters ahead. But mathematics books don't read like novels, so take your time.

If, on the other hand, you like mathematics, it may be that you like only applied mathematics, in which case you won't enjoy this book at all. Though graph theory has many applications (to electrical circuitry, chemistry, industrial management, linear programming, game theory, transportation networks, statistical mechanics, social psychology, and more) we're going to ignore them. After all, the book is supposed to be an antidote for an overdose of applied mathematics.

By this book I hope to influence your present attitude toward mathematics. Many people have a distorted notion of the nature and spirit of mathematics. Whether after reading the book you like or dislike mathematics is beside the point. My goal is rather that you form your attitude toward the subject in response to a clear image of it, not a phantasm.

Suggested reading

**A Mathematician's Apology* by G. H. Hardy (Cambridge University Press, 1967). A once-great has-been's defense of his life's work. I highly recommend it.

*Part II of *Fantasia Mathematica,* edited by Clifton Fadiman (Simon & Schuster, 1958). Part II is a collection of science-fiction stories with mathematical premises.

**Mathematics in Western Culture* by Morris Kline (Oxford University Press, 1964). Kline argues that mathematics has been a major cultural force in the West, influencing not only science and philosophy, but religion, politics, painting, music, and literature as well.

**The Nature and Growth of Modern Mathematics* (2 vols.) by Edna E. Kramer (Fawcett, 1970).

**Philosophy of Mathematics* by Stephen F. Barker (Prentice-Hall, 1964).

**Laws of Form* by G. Spencer Brown (Bantam, 1973). Heavy going; brilliant.

2. GRAPHS

Introduction

The things in Figure 1 are called “graphs”, and are typical of what we will be playing with. Despite the name, they are unrelated to the pictures of equations drawn in high school algebra courses. To quickly convince you of this, I mention that we will consider graphs a), b), and c) to be identical (the word we will use is “isomorphic”), though if drawn in cartesian coordinates they would depict three different equations. In order to define “graph” more precisely, we must first discuss “sets”.

Sets

Definition 1. A *set* is a collection of distinct objects, none of which is the set itself.

If you’ve encountered sets before you may find the last part of the definition a bit puzzling. We’ll return to it in the next section, but for now suffice it to say that we intend to exclude collections like $A = \{1, 2, 3, A\}$.

The words “collection” and “object” are left undefined. Strictly speaking, therefore, we don’t know what we’re talking about. That’s the way it is with pure mathematics. But there is implied an interpretation of “set” as being a bunch of things, an aggregate of entities, etc., and if your intuition feels more comfortable with something to hold onto (mine does), feel free to think of a set as being just what it sounds like. But be careful, for the words “collection” and “object” are used in a slightly unconventional fashion. We are allowing “collections” of infinitely many things, or just one thing, or even no things. And an “object” need not be something we can smell or trip over; to a mathematician an “object” is anything conceivable, including numbers, unicorns, and Peter Pan. The only things left out are the inconceivable, for instance a figure which simultaneously has all the geometric properties of both a triangle and a circle. Everything else is acceptable.

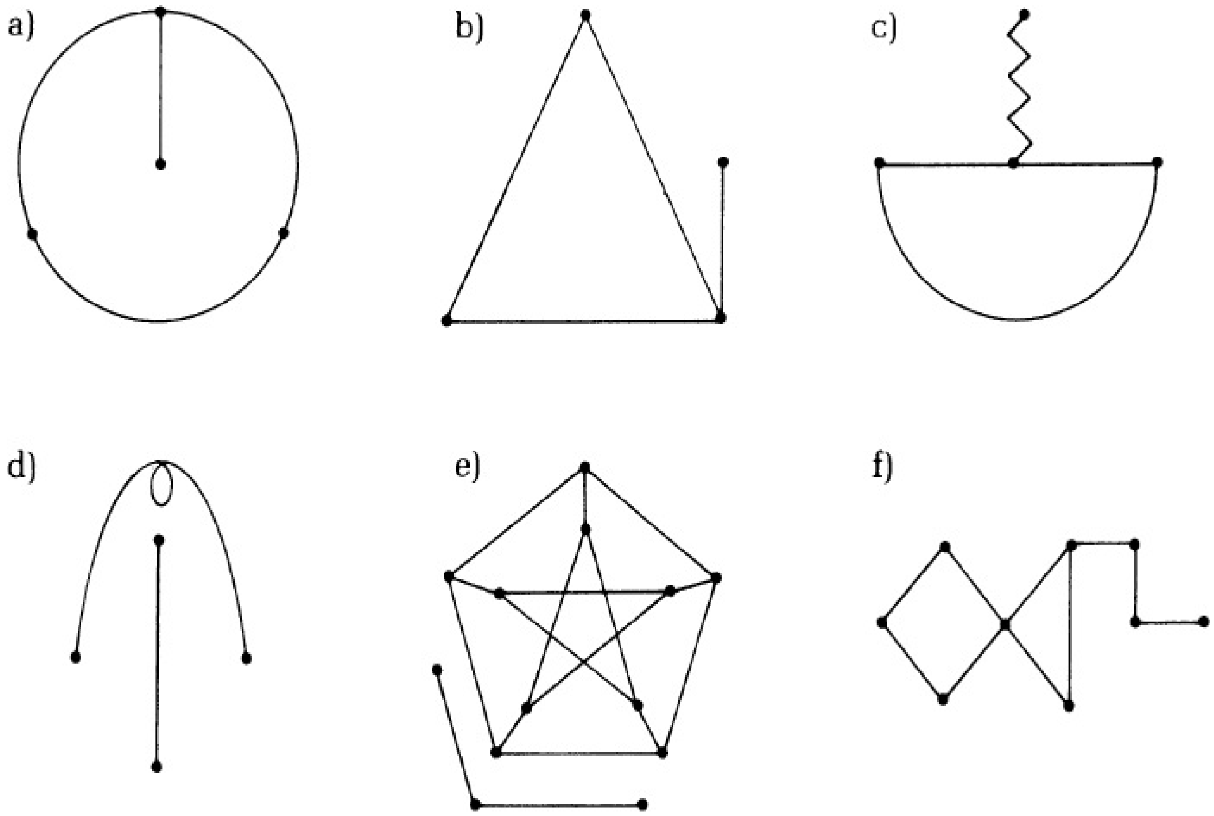

Figure 1

A set is usually denoted by listing the names of the objects, separated by commas, within a pair of braces. So

{12, q, \$, Empire State Building, 9, King Arthur}

is a set. Notice that the objects, which are called the elements of the set, need not have anything in common; mathematical sets are not like sets of luggage. Usually we give sets names to avoid a lot of writing when we subsequently refer to them. Thus one might say, "Let A = {12, q, \$, Empire State Building, 9, King Arthur}." This is the beginning of mathematical notation, something which, like any other shorthand, can be more confusing than helpful if not learned gradually as it comes along. Definitions are just another version of the same thing. I recommend that you memorize (yes!) the meanings of all new terms and symbols when they first appear.

Definition 2. A set containing no elements is called a *null set* or an *empty set*.

An empty set plays the same role in set theory that 0 plays in arithmetic, but you must be careful not to confuse the two. An empty set is a set; 0 is a number. They are of completely different species. Think of it this way: 0 is a number that tells you how many objects there are in an empty set.

Definition 3. *A* set A is said to be a subset of a set B, denoted "$A \subset B$", if every element of A is also an element of B.

Example. If $A = \{1, 2, 3\}$ and $B = \{\&, 3, +, 1, 2\}$, then $A \subset B$, $A \subset A$, and $B \subset B$.

Notice that every set is a subset of itself.

Convention. We agree to consider an empty set to be a subset of every set.

This "convention" can in fact be proved, but only by a logical contortion that seems inappropriate at the present. Thus it is a "convention" instead of a "theorem".

Example. If J is an empty set and A and B are the sets of the previous example, then $J \subset A$, $J \subset B$, and $J \subset J$.

Definition 4. *A* set A is said to be equal to a set B, denoted "$A = B$", if $A \subset B$ and $B \subset A$.

Thus $A = B$ if A and B consist of exactly the same objects. It follows that the order in which the elements of a set appear is irrelevant. For example, $\{1, 2, 3\} = \{2, 1, 3\}$ since $\{1, 2, 3\} \subset \{2, 1, 3\}$ and $\{2, 1, 3\} \subset \{1, 2, 3\}$.

Theorem 1. *There is only one empty set.*

Proof. Let J and K be empty sets. Then by the Convention, $J \subset K$ and $K \subset J$, so $J = K$ by Definition 4. Thus all empty sets are equal; that is, there is really only one empty set.

Notation. The empty set (we had to say "an empty set" before, but now we can say "the empty set") shall be denoted by "$\emptyset$" or "{ }".

Use either notation at your whim, but be careful not to combine them. "$\{\emptyset\}$" would be interpreted by a mathematician as signifying a set containing one element, that element being the symbol $\emptyset$.

In view of the word "distinct" in Definition 1, collections like {1, 3, 3, &, 407}, in which an object is listed more than once, are not sets. The definition was designed to exclude this sort of thing because "{1, 3, 3, &, 407}" is a redundancy. We are being told twice that the collection contains the number 3.

All the set theory we'll need to talk about graphs has now been developed. But before going on I want to take time out and explain why in pure mathematics we are so excruciatingly careful about definitions and proofs (something you may have already noticed). In the process I'll explain the reason for that last clause in Definition 1. You may find parts of the next section a little involved, but I think my point is important and I encourage you to persist.

Paradox

The story begins in the sixth century B.C. with the Pythagoreans, a community of men and women founded by Pythagoras at Crotona in southern Italy. The Pythagoreans shared quasi-religious rituals, dietary laws, and devotion to mathematics as the key to understanding nature. It is they who first discovered that intuition and