THE IMPLICIT FUNCTION THEOREM

HISTORY, THEORY, AND APPLICATIONS

Steven G. Krantz
Harold R. Parks

Let $\psi(z)$ and $\phi(z)$ be analytic on the open disc $D(a,r) \subset \mathbb{C}$ and con
the closed disc $\overline{D}(a,r)$. If t is of small enough modulus that

$$|t \quad (z)| < |z - a|$$

holds for $z \subset \partial D(a,r)$, then

$$\zeta = a + t\phi(\zeta)$$

has exactly one root in $D(a,r)$ and if that root $\zeta = \zeta(i)$ is considered
function of t, then we have

$$\psi(\zeta) = \psi(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left(\frac{d^{n-1}}{dz^{n-1}} [\psi'(z)[\phi(z)]^n] \right)$$

Springer Science+Business Media, LLC

Steven G. Krantz
Harold R. Parks

The Implicit Function Theorem

History, Theory, and Applications

Springer Science+Business Media, LLC

Steven G. Krantz
Department of Mathematics
Washington University
St. Louis, MO 63130-4899
U.S.A.

Harold R. Parks
Department of Mathematics
Oregon State University
Corvallis, OR 97331-4605
U.S.A.

Library of Congress Cataloging-in-Publication Data

Krantz, Steven G. (Steven George), 1951-
The implicit function theorem : history, theory, and applications / Steven G. Krantz and Harold R. Parks.
p. cm.
Includes bibliographical references and index.
ISBN 978-1-4612-6593-1 ISBN 978-1-4612-0059-8 (eBook)
DOI 10.1007/978-1-4612-0059-8
1. Implicit functions. I. Parks, Harold R., 1949- II. Title.

QA331.5.K7136 2002
515'.8–dc21 2002018234
CIP

AMS Subject Classifications: Primary: 26B10; Secondary: 01-01, 01-02, 53B99, 53C99, 35A99, 35B99

Printed on acid-free paper.

Originally published by Birkhäuser Boston in 2003 in 2003
Softcover reprint of the hardcover 1st edition 2003
second printing

ISBN 978-1-4612-6593-1 SPIN 10938065

Reformatted from authors' files by TEXniques, Inc., Cambridge, MA.

9 8 7 6 5 4 3 2

To the memory of Kennan Tayler Smith (1926–2000)

Contents

Preface

The implicit function theorem is, along with its close cousin the inverse function theorem, one of the most important, and one of the oldest, paradigms in modern mathematics. One can see the germ of the idea for the implicit function theorem in the writings of Isaac Newton (1642–1727), and Gottfried Leibniz's (1646–1716) work explicitly contains an instance of implicit differentiation. While Joseph Louis Lagrange (1736–1813) found a theorem that is essentially a version of the inverse function theorem, it was Augustin-Louis Cauchy (1789–1857) who approached the implicit function theorem with mathematical rigor and it is he who is generally acknowledged as the discoverer of the theorem. In Chapter 2, we will give details of the contributions of Newton, Lagrange, and Cauchy to the development of the implicit function theorem.

The form of the implicit function theorem has evolved. The theorem first was formulated in terms of complex analysis and complex power series. As interest in, and understanding of, real analysis grew, the real-variable form of the theorem emerged. First the implicit function theorem was formulated for functions of two real variables, and the hypothesis corresponding to the Jacobian matrix being nonsingular was simply that one partial derivative was nonvanishing. Finally, Ulisse Dini (1845–1918) generalized the real-variable version of the implicit function theorem to the context of functions of any number of real variables. As mathematicians understood the theorem better, alternative proofs emerged, and the associated modern techniques have allowed a wealth of generalizations of the implicit function theorem to be developed.

Today we understand the implicit function theorem to be an *ansatz*, or a way of looking at problems. There are implicit function theorems, inverse function theorems, rank theorems, and many other variants. These theorems are valid on

Euclidean spaces, manifolds, Banach spaces, and even more general settings. Roughly speaking, the implicit function theorem is a device for solving equations, and these equations can live in many different settings.

In addition, the theorem is valid in many categories. The textbook formulation of the implicit function theorem is for C^k functions. But in fact the result is true for $C^{k,\alpha}$ functions, Lipschitz functions, real analytic functions, holomorphic functions, functions in Gevrey classes, and for many other classes as well. The literature is rather opaque when it comes to these important variants, and a part of the present work will be to set the record straight.

Certainly one of the most powerful forms of the implicit function theorem is that which is attributed to John Nash (1928–) and Jürgen Moser (1928–1999). This device is actually an infinite iteration scheme of implicit function theorems. It was first used by John Nash to prove his celebrated imbedding theorem for Riemannian manifolds. Jürgen Moser isolated the technique and turned it into a powerful tool that is now part of partial differential equations, functional analysis, several complex variables, and many other fields as well. This text will culminate with a version of the Nash–Moser theorem, complete with proof.

This book is one both of theory and practice. We intend to present a great many variants of the implicit function theorem, complete with proofs. Even the important implicit function theorem for real analytic functions is rather difficult to pry out of the literature. We intend this book to be a convenient reference for all such questions, but we also intend to provide a compendium of examples and of techniques. There are applications to algebra, differential geometry, manifold theory, differential topology, functional analysis, fixed point theory, partial differential equations, and to many other branches of mathematics. One learns mathematics (in part) by watching others do it. We hope to set a suitable example for those wishing to learn the implicit function theorem.

The book should be of interest to advanced undergraduates, graduate students, and professional mathematicians. Prerequisites are few. It is not necessary that the reader be already acquainted with the implicit function theorem. Indeed, the first chapter provides motivation and examples that should make clear the form and function of the implicit function theorem. A bit of knowledge of multivariable calculus will allow the reader to tackle the elementary proofs of the implicit function theorem given in Chapter 3. Rudiments of real and functional analysis are needed for the third proof in Chapter 3 which uses the Contraction Mapping Fixed Point Principle. Some knowledge of complex analysis is required for a complete reading of the historical material—this seems to be unavoidable since the earliest rigorous work on the implicit function theorem was formulated in the context of complex variables. In many cases a willing suspension of disbelief and a bit of determination will serve as a thorough grounding in the basics.

There are many sophisticated applications of implicit function theorems, particularly the Nash–Moser theorem, in modern mathematics. The imbedding theorem for Riemannian manifolds, the imbedding theorem for CR manifolds, and the deformation theory of complex structures are just a few of them. Richard Hamilton's masterful survey paper (see the Bibliography) indicates several more applications